PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 7, July 1995

COMPOSITION OPERATORS BETWEEN HARDY AND WEIGHTED BERGMAN SPACES ON CONVEX DOMAINS IN $\mathbf{C}^N$

BARBARA D. MACCLUER AND PETER R. MERCER

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Suppose Ω is a bounded, strongly convex domain in $\mathbf{C}^N$ with smooth boundary and $\phi\colon \Omega \to \Omega$ is an arbitrary holomorphic map. While in general the composition operator C_ϕ need not map the Hardy space $H^p(\Omega)$ into itself when $N > 1$, our main theorem shows that C_ϕ does map $H^p(\Omega)$ boundedly into a certain weighted Bergman space on Ω, where the weight function depends on the dimension N. We also consider properties of C_ϕ on $H^p(\Omega)$ when $\phi(\Omega)$ is contained in an approach region in Ω.

1. INTRODUCTION

For Ω a domain in $\mathbf{C}^N$ and $\phi\colon \Omega \to \Omega$ holomorphic, the composition operator C_ϕ with symbol ϕ is defined by $C_\phi(f) = f \circ \phi$, for f holomorphic on Ω. When Ω is the unit disc Δ in $\mathbf{C}$, it is well known that for every holomorphic $\phi\colon \Delta \to \Delta$, C_ϕ will be a bounded operator on the Hardy spaces $H^p(\Delta)$, for all p. However when Ω is the unit ball B_N, $N > 1$, this is no longer the case; various examples have been given to show that C_ϕ may fail to be bounded on the Hardy spaces $H^p(B_N)$ $(0 < p < \infty)$, and several authors [CW, M, MS, W] have considered the problem of characterizing those ϕ for which C_ϕ is bounded on $H^p(B_N)$. A completely satisfactory answer to this question is not yet known.

Here we will consider more generally the case that Ω is a bounded, strongly convex domain in $\mathbf{C}^N$ with smooth boundary. Our main result will show that for *every* holomorphic $\phi\colon \Omega \to \Omega$, C_ϕ is a bounded map of $H^p(\Omega)$ into the weighted Bergman space $A^{p,\alpha}(\Omega) \equiv \{f \text{ holomorphic} : \int_\Omega |f(z)|^p \, d\lambda_\Omega^\alpha(z) < \infty\}$, where $\alpha = N-2$ and $d\lambda_\Omega^\alpha(z) = (\operatorname{dist}(z, \partial\Omega))^\alpha \, d\lambda_\Omega(z)$ with $d\lambda_\Omega(z)$ normalized volume measure on Ω and $\operatorname{dist}(z, \partial(\Omega))$ the Euclidean distance from z to $\partial\Omega$. In particular, when $\Omega = B_N$, this says that every C_ϕ maps $H^p(B_N)$ boundedly into $A^{p,N-2}(B_N) = \{f \text{ holomorphic:} \int_{B_N} |f(z)|^p (1-|z|)^{N-2} \, d\lambda_B(z)\}$.

Received by the editors October 15, 1993.
1991 *Mathematics Subject Classification*. Primary 47B38; Secondary 32H02.
The first author's research was supported in part by a grant from the National Science Foundation.
The second author's research was supported in part by N.S.E.R.C. Canada.

We also consider compactness properties of $C_\phi: H^p(\Omega) \to H^p(\Omega)$ when $\phi(\Omega)$ is contained in an approach region at some $\zeta \in \partial\Omega$. Our result here generalizes a previously known result in the ball ([M, Theorem 2.2]) which showed that if $\phi(B_N)$ is contained in a Koranyi approach region of appropriately small aperture (depending on the dimension N), then C_ϕ is compact as an operator fron $H^p(B_N)$ to $H^p(B_N)$.

2. Notation and background

In this section we fix our notation and collect some relevant background material. In all of what follows, $\Omega \subset\subset \mathbf{C}^N$ will be a bounded strongly convex domain with C^∞ boundary, and B_N will be the unit ball in $\mathbf{C}^N$, except that when $N = 1$ we write instead Δ for the unit disc. Normalized surface area measure on $\partial\Omega$ and ∂B_N will be denoted by σ_Ω and σ_{B_N} (or just σ_B if the dimension need not be explicitly shown) respectively. Similarly λ_Ω, λ_B denote normalized volume measure on Ω and B_N. Recall that a holomorphic map from Δ to Ω is called an extremal map ([L1]), or complex geodesic ([V]) if it is an isometry with respect to the Kobayashi distances on Δ and Ω. For $z_0 \in \Omega$ and $x \in \partial\Omega$ there is a unique extremal map $\varphi_x: \overline{\Delta} \to \overline{\Omega}$ satisfying $\varphi_x(0) = z_0$ and $\varphi_x(1) = x$; φ_x is C^∞ on $\overline{\Delta}$ with $\varphi_x(\partial\Delta) \subset \partial\Omega$ ([L1, A]).

Associated with each such extremal map φ_x is a retraction p_x, which is a holomorphic map of Ω onto $\varphi_x(\Delta) \subset \Omega$ satisfying

$$p_x \circ \varphi_x(\lambda) = \varphi_x(\lambda) \quad \forall \lambda \in \Delta,$$

and

$$p_x \circ p_x = p_x$$

[L1, RW].

Note that in the ball B_N with $z_0 = 0$ we have $\varphi_x(\lambda) = \lambda x$ $(x \in \partial B_N, \lambda \in \Delta)$ and $p_x(z) = \langle z, x\rangle x$ $(z \in B_N)$, where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbf{C}^N$.

There is also, for each fixed direction $v \neq 0$ in $\mathbf{C}^N$ and $z_0 \in \Omega$, a unique extremal map $\varphi_v: \overline{\Delta} \to \overline{\Omega}$, C^∞ on $\partial\Delta$, satisfying

$$\varphi_v(0) = z_0, \\ \varphi_v'(0) = rv$$

with $r > 0$ maximal [L1, RW]. From these extremal maps Lempert has constructed a canonical mapping $\Psi: \overline{B}_N \to \overline{\Omega}$ called the spherical representation of Ω, defined (for z_0 fixed in Ω) by

$$\Psi(0) = z_0, \\ \Psi(v) = \varphi_v(\|v\|), \qquad v \neq 0 \text{ in } \overline{B}_N.$$

It is easy to see that Ψ is holomorphic on slices through the origin. Moreover, Ψ is a homeomorphism between $\overline{B_N}$ and $\overline{\Omega}$ which is a smooth diffeomorphism off any neighborhood of 0 (but including ∂B_N) [L1, Théorème 3; L3, Theorem 5.1]. Its inverse $\Psi^{-1}: \overline{\Omega} \to \overline{B}_N$ is given by

$$\Psi^{-1}(z) = \tanh k_\Omega(z_0, z)\frac{\varphi_z'(0)}{\|\varphi_z'(0)\|},$$

where k_Ω is the Kobayashi distance on Ω and $\tanh k_\Omega(z_0, z) = 1$ if $z \in \partial\Omega$. Moreover, $\varphi_x(re^{i\theta}) = \Psi(re^{i\theta}\Psi^{-1}(x))$ for $r \in [0, 1]$, θ real, and $x \in \partial\Omega$ ([A]). The reader may check that with $\Omega = B_N$ and $z_0 = 0$, Ψ is the identity.

The map Ψ allows us to prove a version of slice integration on $\partial\Omega$, generalizing Proposition 1.4.7(i) of [R] for the ball. This lemma will be used in §4.

Lemma 1. *For integrable f on $\partial\Omega$*

$$\int_{\partial\Omega} f\,d\sigma_\Omega \approx \int_{\partial\Omega}\int_0^{2\pi} f\circ\varphi_x(e^{i\theta})\,d\theta\,d\sigma_\Omega(x),$$

where $\approx$ indicates that the ratio of the two quantities is bounded above and below by finite positive constants independent of f.

Proof. Fix $z_0 \in \Omega$ and let Ψ be the spherical representation of Ω. Use the fact that Ψ is a diffeomorphism between ∂B_N and $\partial\Omega$ together with slice integration in B_N to see that

$$\begin{aligned}\int_{\partial\Omega} f\,d\sigma_\Omega &= \int_{\partial B_N} f\circ\Psi\,d\sigma_\Omega\circ\Psi \approx \int_{\partial B_N} f\circ\Psi\,d\sigma_B\\ &= \frac{1}{2\pi}\int_{\partial B_N}\int_0^{2\pi} f\circ\Psi(e^{i\theta}\eta)\,d\theta\,d\sigma_B(\eta)\\ &= \frac{1}{2\pi}\int_{\partial\Omega}\int_0^{2\pi} f\circ\Psi(e^{i\theta}\Psi^{-1}(x))\,d\theta\,d\sigma_B\circ\Psi^{-1}(x)\\ &\approx \int_{\partial\Omega}\int_0^{2\pi} f\circ\Psi(e^{i\theta}\Psi^{-1}(x))\,d\theta\,d\sigma_\Omega(x)\\ &= \int_{\partial\Omega}\int_0^{2\pi} f\circ\varphi_x(e^{i\theta})\,d\theta\,d\sigma_\Omega(x). \quad\square\end{aligned}$$

The following lemma due to Lempert will play an essential role in our main theorems in §§3 and 4. Let $d_\Omega(z)$ be the Euclidean distance from z to $\partial\Omega$.

Lemma 2 [L1, Proposition 12]. *Let $\varphi\colon \Delta \to \Omega$ be an extremal map. There exists a finite constant c depending only on $\varphi(0)$ so that*

$$d_\Omega(\varphi(\lambda)) \le c(1 - |\lambda|)$$

for all $\lambda \in \Delta$.

3. Main theorem

For $0 < p < \infty$ the spaces $H^p(\Omega)$ are the usual holomorphic Hardy spaces on Ω ([S; K, Chapter 8]). For $\alpha \ge 0$ define the weighted Bergman space $A^{p,\alpha}(\Omega)$ to be those holomorphic f on Ω satisfying

$$\|f\|_\alpha^p \equiv \int_\Omega |f(z)|^p\,d_\Omega^\alpha(z)\,d\lambda_\Omega(z) < \infty,$$

where $d_\Omega^\alpha(z) = (\operatorname{dist}(z, \partial\Omega))^\alpha$. The weighted volume measure $d_\Omega^\alpha(z)\,d\lambda_\Omega(z)$ will be abbreviated as $d\lambda_\Omega^\alpha$. In the case $\Omega = B_N$ we have the standard weighted Bergman spaces.

Our first lemma is a quantitative formulation of the fact that in the disc every composition operator is bounded on the standard weighted Bergman spaces.

This result is well known (it is a consequence of Proposition 3.4 and Theorem 4.3 of [MS]); we include its proof for completeness. For $e^{i\theta} \in \partial\Delta$ and $t > 0$, $S(e^{i\theta}, t) = \{z \in \Delta : |1 - ze^{-i\theta}| < t\}$, and let α be non-negative.

Lemma 3. *Let $\gamma: \Delta \to \Delta$ be holomorphic with $\gamma(0) = 0$. There exists an absolute constant C (independent of the particular choice of γ) so that*

$$\lambda_\Delta^\alpha(\gamma^{-1}(S(e^{i\theta}, t))) \le Ct^{\alpha+2}$$

for all real θ and $t > 0$.

Proof. Littlewood's subordination principle shows that the operator C_γ is bounded on the weighted Bergman space $A^{p,\alpha}(\Delta)$, with the norm of $C_\gamma = 1$. In particular, for all $f \in A^{2,\alpha}(\Delta)$ we have

$$\int_\Delta |f \circ \gamma|^2 \, d\lambda_\Delta^\alpha(z) \le \int_\Delta |f|^2 \, d\lambda_\Delta^\alpha(z)$$

or

$$\int_\Delta |f|^2 d(\lambda_\Delta^\alpha \gamma^{-1}) \le \int_\Delta |f|^2 \, d\lambda_\Delta^\alpha(z).$$

Apply this to the test functions

$$f_w = (1 - z\overline{w})^{-(\alpha+2)} \in A^{2,\alpha}(\Delta)$$

(which have norms in $A^{2,\alpha}(\Delta)$ comparable to $(1 - |w|^2)^{-(\alpha+2)/2})$ for the choice $w = (1-t)e^{i\theta}$. Using the estimate

$$|f_w(z)|^2 \ge c_1 \frac{1}{(t^{\alpha+2})^2} \quad \text{on } S(e^{i\theta}, t)$$

for some absolute constant c_1 we obtain

$$c_1 \frac{1}{t^{2(\alpha+2)}} \lambda_\Delta^\alpha \gamma^{-1}(S(e^{i\theta}, t)) \le c_2 \frac{1}{t^{\alpha+2}},$$

which gives the desired result. □

Now we return to our map $\phi: \Omega \to \Omega$. Pick an arbitrary base point $z_0 \in \Omega$ and let $\phi(z_0) = w_0$. Fix $\zeta \in \partial\Omega$, and let $\varphi_\zeta: \Delta \to \Omega$ be the extremal map with $\varphi_\zeta(0) = w_0$ and $\varphi_\zeta(1) = \zeta$. Denote its associated retraction by p_ζ. For each $x \in \partial\Omega$ we consider the holomorphic self-map τ_x of the disc Δ defined by

$$\tau_x = \varphi_\zeta^{-1} \circ p_\zeta \circ \phi \circ \varphi_x,$$

where the extremal maps $\varphi_x: \Delta \to \Omega$ have $\varphi_x(0) = z_0$ and $\varphi_x(1) = x$. Notice that for each $x \in \partial\Omega$, $\tau_x(0) = 0$. Also define sets $S(\zeta, t)$ for $t > 0$ by

$$S(\zeta, t) = \{z \in \Omega : |1 - \varphi_\zeta^{-1} \circ p_\zeta(z)| < t\}.$$

In the special case $\Omega = B_N$, $z_0 = w_0 = 0$ the maps τ_x are just

$$\tau_x(\lambda) = \langle \phi(\lambda x), \zeta \rangle$$

and $S(\zeta, t) = \{z \in B_N : |1 - \langle z, \zeta \rangle| < t\}$, which is the usual definition of a Carleson set (based at ζ) in B_N and consistent with our previous use of the notation $S(e^{i\theta}, t)$ in Δ. We will see later that in general these sets $S(\zeta, t)$ are equivalent to the usual Carleson sets in Ω as defined by Hormander.

The next proposition uses the maps τ_x to estimate

$$\lambda_\Omega^\alpha \phi^{-1} S(\zeta, t).$$

This estimate will be the key ingredient in the proof of our main theorem.

Proposition 4. *There exists a finite constant* C, *independent of* $\zeta \in \partial\Omega$ *and* $t > 0$, *so that*

$$\lambda_\Omega^\alpha \phi^{-1} S(\zeta, t) \le C t^{\alpha+2}.$$

In particular, when $\alpha = N - 2$, *we have*

$$\lambda_\Omega^{N-2} \phi^{-1} S(\zeta, t) = O(t^N)$$

for $\zeta \in \partial\Omega$, $t > 0$.

Proof. Clearly it is enough to show this for all $t < t_0$, where t_0 is an arbitrary positive number. Fix a neighborhood V of z_0, $V \subset\subset \Omega$, and find $t_0 > 0$ so that $\phi^{-1}S(\zeta, t) \cap V$ is empty for all $\zeta \in \partial\Omega$ and $t < t_0$. Then find $\varepsilon > 0$ so that if $\Psi: \overline{B_N} \to \overline{\Omega}$ is the spherical representation of Ω described in §2 (with $\Psi(0) = z_0$) we have $\Psi(B_\varepsilon) \subset V$, where B_ε is the ball of radius ε centered at 0.

Writing $\chi_{\phi^{-1}S}$ for the characteristic function of a set $\phi^{-1}S(\zeta, t)$ we have the following estimate for all $0 < t < t_0$:

$$\begin{aligned} \lambda_\Omega^\alpha \phi^{-1} S(\zeta, t) &= \int_\Omega \chi_{\phi^{-1}S}(w) d_\Omega^\alpha(w)\, d\lambda_\Omega(w) \\ &= \int_{\Omega\setminus V} \chi_{\phi^{-1}S}(w)\, d_\Omega^\alpha(w)\, d\lambda_\Omega(w) \\ &\le c_1 \int_{B\setminus B_\varepsilon} \chi_{\phi^{-1}S} \circ \Psi(z)\, d_\Omega^\alpha(\Psi(z))\, d\lambda_B(z), \end{aligned}$$

where c_1 is a constant independent of ζ and t, since Ψ is a diffeomorphism on $\overline{B_N \setminus B_\varepsilon}$. Changing to polar coordinates and using slice integration in the ball gives that this last integral is

$$\begin{aligned} & c_2 \int_\varepsilon^1 r^{2N-1} \int_{\partial B} \int_0^{2\pi} \chi_{\phi^{-1}S} \circ \Psi(re^{i\theta}\eta)\, d_\Omega^\alpha(\Psi(re^{i\theta}\eta))\, d\theta\, d\sigma_B(\eta)\, dr \\ &= c_2 \int_\varepsilon^1 r^{2N-1} \int_{\partial\Omega} \int_0^{2\pi} \chi_{\phi^{-1}S} \circ \Psi(re^{i\theta}\Psi^{-1}(x)) \\ &\qquad \cdot d_\Omega^\alpha(\Psi(re^{i\theta}\Psi^{-1}(x))\, d\theta\, d\sigma_B \circ \Psi^{-1}(x)\, dr \\ &\le c_3 \int_\varepsilon^1 r^{2N-1} \int_{\partial\Omega} \int_0^{2\pi} \chi_{\phi^{-1}S} \circ \Psi(re^{i\theta}\Psi^{-1}(x)) \\ &\qquad \cdot d_\Omega^\alpha(\Psi(re^{i\theta}\Psi^{-1}(x))\, d\theta\, d\sigma_\Omega(x)\, dr, \end{aligned}$$

again because Ψ is a diffeomorphism on ∂B_N. Since

$$\Psi(re^{i\theta}\Psi^{-1}(x)) = \varphi_x(re^{i\theta}),$$

the integrand in the inner integral of the last line is $\chi_{\phi^{-1}S}(\varphi_x(re^{i\theta}))\, d_\Omega^\alpha(\varphi_x(re^{i\theta}))$, which, by Lemma 2, is bounded above by a constant multiple of $\chi_{\phi^{-1}S}(\varphi_x(re^{i\theta})) \cdot (1-r)^\alpha$ (where we use the fact that $\varphi_x(0) = z_0$ for all $x \in \partial\Omega$). At this point we have

$$\begin{aligned} \lambda_\Omega^\alpha \phi^{-1} S(\zeta, t) &\le c_4 \int_\varepsilon^1 r^{2N-1} \int_{\partial\Omega} \int_0^{2\pi} \chi_{\phi^{-1}S}(\varphi_x(re^{i\theta}))\, d\theta\, d\sigma_\Omega(x) (1-r)^\alpha\, dr \\ &\le c_4 \int_{\partial\Omega} \int_\Delta \chi_{\phi^{-1}S} \varphi_x(u)\, d\lambda_\Delta^\alpha(u)\, d\sigma_\Omega(x). \end{aligned}$$

Now

$$\begin{aligned}\chi_{\phi^{-1}S}\varphi_x(u) = 1 &\Leftrightarrow \phi \circ \varphi_x(u) \in S(\zeta, t) \\ &\Leftrightarrow |1 - \varphi_\zeta^{-1} \circ p_\zeta \circ \phi \circ \varphi_x(u)| < t \\ &\Leftrightarrow \tau_x(u) \in S(1, t).\end{aligned}$$

Applying Lemma 3 gives

$$\lambda_\Omega^\alpha \phi^{-1} S(\zeta, t) \le c_5 t^{\alpha+2},$$

where c_5 depends on neither ζ nor t, and we are done. □

To use this proposition to prove the main theorem we need to relate the sets $S(\zeta, t)$ to the Carleson sets defined by Hormander [H] (in the general setting of bounded strictly pseudoconvex domains with C^2 boundary). These sets, denoted $A(\zeta, t)$ for $\zeta \in \partial\Omega$ and $t > 0$, are defined as follows: Let π_ζ be the complex tangent space at ζ, and let $A(\zeta, t)$ be all points in Ω whose distance to the ball in π_ζ with center ζ and radius $\sqrt{t}$ is at most t. We claim that the sets $S(\zeta, t)$ and $A(\zeta, t)$ are comparable in the sense that there exist finite positive constants k_1, k_2 so that

$$A(\zeta, k_2 t) \subseteq S(\zeta, t) \subseteq A(\zeta, k_1 t).$$

To see this we make use of a special biholomorphic map of Ω to a domain Ω' which flattens out the image of the extremal disc $\varphi_\zeta(\Delta)$ in Ω. The existence of this biholomorphism and the relevant properties of it and the domain Ω' are contained in the following theorem.

Theroem 5 [L1, L2, L4]. *Given a strongly convex domain $\Omega \subset\subset \mathbf{C}^N$ with C^∞ boundary and an extremal map $\varphi: \Delta \to \Omega$ with associated retraction p, there exists a domain $\Omega' \subset\subset \mathbf{C}^N$ and a biholomorphism $\Lambda: \Omega \to \Omega'$ which is C^∞ on $\overline{\Omega}$ such that*:

(i) $\Lambda \circ \varphi(\lambda) = (\lambda, 0')$ $\forall \lambda \in \Delta$.
(ii) $\Lambda \circ p = \pi \circ \Lambda$, *where* $\pi(z_1, z') = (z_1, 0')$.
(iii) *For each* $\xi \in \partial\Delta$ *we have* $(\xi, 0') \in \partial\Omega'$ *with the unit outward normal there* $(\xi, 0')$.
(iv) Ω' *is strongly convex in a neighborhood of* $\{(\xi, 0'): |\xi| = 1\}$.

Lemma 6. *The sets $S(\zeta, t)$ and $A(\zeta, t)$ are comparable.*

Proof. The sets $A(\zeta, t)$ are invariant under biholomorphic maps ([H, p. 73]), so it suffices to show that the sets $\Lambda(S(\zeta, t))$ and $A(\Lambda(\zeta), t)$ are comparable, where $\zeta \in \partial\Omega$. We use Theorem 5, with $\varphi_\zeta = \varphi$ and $p_\zeta = p$. By (i) and (ii) we have $\Lambda(S(\zeta, t)) = \{z \in \Omega': |1 - z_1| < t\}$. By (iii) and (iv) and the definition of $A(\zeta, t)$ there are constants $k_1, k_2 > 0$ such that

$$A(\Lambda(\zeta), k_2 t) \subset \Lambda(S(\zeta, t)) \subset A(\Lambda(\zeta), k_1 t) \quad (\forall t > 0).$$

Now the sets $\varphi_\zeta(\Delta)$ vary continuously with $\zeta \in \partial\Omega$ ([L1, Proposition 11; A, Lemma 1.9]), so we may assume that k_1, k_2 are independent of ζ, and we are done. □

Our main theorem is as follows.

Theorem 7. *If Ω is a bounded, strongly convex domain in $\mathbf{C}^N$ $(N \geq 2)$ with C^∞ boundary, and if $\phi\colon \Omega \to \Omega$ is homomorphic, then C_ϕ maps $H^p(\Omega)$ boundedly into $A^{p, N-2}(\Omega)$, for each $0 < p < \infty$.*

Proof. We wish to show that there exists $C < \infty$ so that

$$\int_\Omega |f \circ \phi(z)|^p \, d\lambda_\Omega^{N-2}(z) \leq C \int_{\partial\Omega} |f|^p \, d\sigma_\Omega ,$$

whenever $f \in H^p(\Omega)$. By Hormander's Carleson measure theorem ([H, Theorem 4.3]) it suffices to show that there exists $C' < \infty$ satisfying

$$(*) \qquad \lambda_\Omega^{N-2} \phi^{-1} A(\zeta, t) \leq C' t^N$$

for all $\zeta \in \partial\Omega$, $t > 0$. Lemma 6 shows that we may replace $A(\zeta, t)$ in $(*)$ by $S(\zeta, t)$, and then the result follows from Proposition 4. □

4. Maps into admissible regions

In Theorem 2.2 of [M] it was shown that if the holomorphic map $\phi\colon B_N \to B_N$ has $\phi(B_N)$ contained in a Koranyi approach region of sufficiently small aperture (depending on the dimension N), then C_ϕ will be compact on $H^p(B_N)$; moreover, this result is sharp in a natural sense. In this section we give a theorem in the same spirit for holomorphic maps of a bounded strongly convex domain Ω with C^∞ boundary. As the ideas are similar to those in the last section we will omit some details of the arguments.

Analogous to Lemma 3 in §3 we begin with a one variable result, which considers maps which take Δ into a non-tangential approach region in Δ.

Lemma 8 [M, Lemma 2.3]. *If $\gamma\colon \Delta \to \Delta$ is holomorphic with $\gamma(0) = 0$ and $\gamma(\Delta) \subseteq \{z \in \Delta\colon |1 - z| < \alpha(1 - |z|)\}$, then there exists $C < \infty$ depending only on α so that*

$$\sigma_\Delta(\gamma^{-1} S(1, t)) \leq C t^b ,$$

where $b = \frac{\pi}{2\cos^{-1}(1/\alpha)}$. In particular C_γ will be compact (for all $\alpha > 1$).

Recall that if we fix a base point z_0 in our strongly convex domain Ω and consider any extremal map $\varphi\colon \Delta \to \Omega$ with $\varphi(0) = z_0$, then Lemma 2 guarantees that $d_\Omega(\varphi(\lambda)) \leq c_1(1 - |\lambda|)$ for all $\lambda \in \Delta$, where c_1 is a finite constant not depending on a particular map φ. By Theorem 5 there is a positive constant c_2 such that $d_\Omega(z) \leq c_2 d_\Omega(p(z))$ for every $z \in \Omega$, where p is the retraction associated with φ. Just as in the proof of Lemma 6, c_2 is independent of $\varphi(1)$. Thus

$$(4.1) \qquad d_\Omega(z) \leq c_1 c_2 (1 - |\varphi^{-1} \circ p(z)|)$$

for all z in Ω. For $\zeta \in \partial\Omega$, define approach regions $D_\alpha(\zeta)$ by

$$D_\alpha(\zeta) = \{z \in \Omega\colon |1 - \varphi_\zeta^{-1} \circ p_\zeta(z)| < \alpha d_\Omega(z)\}.$$

Notice that by (4.1) $D_\alpha(\zeta)$ is empty for $\alpha < \frac{1}{c_1 c_2}$.

Theorem 9. *Let Ω be a smooth, strongly convex bounded domain. There exist positive constants α_0, α_1 (which depend on Ω, a chosen base point z_0 in Ω, and for α_1 explicitly on the dimension N) so that for each $\zeta \in \partial\Omega$ we have:*

(1) *$D_\alpha(\zeta)$ is empty for $\alpha < \alpha_0$.*

(2) *If* $\phi(\Omega) \subset D_\alpha(\zeta)$ *for* $\alpha_0 < \alpha < \alpha_1$, *then* C_ϕ *is compact from* $H^p(\Omega)$ *to* $H^p(\Omega)$, $0 < p < \infty$.
(3) *If* $\phi(\Omega) \subset D_{\alpha_1}(\zeta)$, *then* C_ϕ *is bounded from* $H^p(\Omega)$ *to* $H^p(\Omega)$, $0 < p < \infty$.

Sketch of proof. We have already observed (1) holds when $\alpha_0 = \frac{1}{c_1 c_2}$. By a recent theorem of Li and Russo [LR] and Lemma 6, C_ϕ is compact from $H^p(\Omega)$ to $H^p(\Omega)$ if

$$\sigma_\Omega \phi^{-1}\overline{S}(\eta, t) = o(t^N) \tag{4.2}$$

as $t \to 0$, uniformly in $\eta \in \partial\Omega$; and C_ϕ is bounded from $H^p(\Omega)$ to $H^p(\Omega)$ when

$$\sigma_\Omega \phi^{-1}\overline{S}(\eta, t) = O(t^N) \quad \forall t > 0, \eta \in \partial\Omega. \tag{4.3}$$

Thus (2) follows if $\alpha \in (\alpha_0, \alpha_1)$ and $\phi(\Omega) \subset D_\alpha(\zeta) \Rightarrow \sigma_\Omega \phi^{-1}\overline{S}(\eta, t) = o(t^N)$, and (3) follows if $\phi(\Omega) \subset D_{\alpha_1}(\zeta) \Rightarrow \sigma_\Omega \phi^{-1}\overline{S}(\eta, t) = O(t^N)$. Here we are identifying the map ϕ with its extension a.e. to $\partial\Omega$ along inner normal vectors ([K, Proposition 8.5.1]). This was also implicit in Lemma 8.

We first consider the process of estimating $\sigma_\Omega \phi^{-1}\overline{S}(\eta, t)$ when $\eta = \zeta$ and $\phi(\Omega) \subseteq D_\alpha(\zeta)$. Exactly as in the proof of Proposition 4 define maps $\tau_x : \Delta \to \Delta$ ($x \in \partial\Omega$) by $\tau_x = \varphi_\zeta^{-1} \circ p_\zeta \circ \phi \circ \varphi_x$. We have $\tau_x(0) = 0$, and by hypothesis and (4.1),

$$|1 - \tau_x(\lambda)| \le c_1 c_2 \alpha (1 - |\varphi_\zeta^{-1} \circ p_\zeta \circ \phi \circ \varphi_x(\lambda)|)$$

which says $\tau_x(\Delta) \subseteq \{\lambda \in \Delta : |1 - \lambda| \le c_1 c_2 \alpha(1 - |\lambda|)\}$. By Lemma 8

$$\sigma_\Delta \tau_x^{-1} S(1, t) \le C t^b,$$

where C depends on $c_1 c_2 \alpha$ and $b = \pi/(2\cos^{-1}(\frac{1}{c_1 c_2 \alpha}))$.

By Lemma 1,

$$\begin{aligned}\sigma_\Omega(\phi^{-1}(\overline{S}(\zeta, t) \cap \partial\Omega) &\equiv \sigma_\Omega(A) \\ &\le c \int_{\partial\Omega} \int_0^{2\pi} \chi_A \circ \varphi_x(e^{i\theta})\, d\theta\, d\sigma_\Omega(x) \\ &\le c t^b,\end{aligned}$$

since $\chi_A \circ \varphi_x(e^{i\theta}) = 1 \Leftrightarrow |1 - \tau_x(e^{i\theta})| \le t$.

Now set $\alpha_1 = \frac{1}{c_1 c_2} \sec \frac{\pi}{2N}$. If $\phi(\Omega) \subseteq D_\alpha(\zeta)$ where $\alpha < \alpha_1$, then $b > N$ and the above calculation shows $\sigma_\Omega(\phi^{-1}\overline{S}(\zeta, t) \cap \partial\Omega) = o(t^N)$ while if $\phi(\Omega) \subseteq D_{\alpha_1}(\zeta)$ we have

$$\sigma_\Omega(\phi^{-1}\overline{S}(\zeta, t) \cap \partial\Omega) = O(t^N).$$

To obtain these same estimates for

$$\sigma_\Omega(\phi^{-1}\overline{S}(\eta, t) \cap \partial\Omega)$$

where $\eta \in \partial\Omega$ is arbitrary, we observe the following: there is a constant k_1, independent of η and t, so that if $S(\eta, t) \cap D_\alpha(\zeta)$ is non-empty, then $S(\zeta, k_1 t) \supseteq S(\eta, t)$. To see this first notice that there is a constant k_2 (depending only on Ω) so that if $S(\eta, t)$ and $S(\zeta, t)$ intersect, then $S(\zeta, k_2 t) \supseteq S(\eta, t)$ (see, for example, [H, p. 73] where the analogous property is proved for the comparable sets $A(\zeta, t))$. Now observe that if $z \in S(\eta, t) \cap D_\alpha(\zeta)$, then $z \in S(\zeta, k_3\alpha t)$ for some constant k_3 and therefore $S(\eta, t) \subseteq S(\zeta, k_2k_3\alpha t) \equiv S(\zeta, k_1 t)$. Thus

$$\begin{aligned}\sigma_\Omega(\phi^{-1}\overline{S}(\eta, t) \cap \partial\Omega) &\leq \sigma_\Omega(\phi^{-1}\overline{S}(\zeta, k_1 t) \cap \partial\Omega) \\ &\leq c(k_1 t)^b = c' t^b,\end{aligned}$$

with b as before, as desired. □

In the case $\Omega = B_N$ and $z_0 = 0$ both c_1 and c_2 can be taken to be 1 and the regions $D_\alpha(\zeta)$ are the usual Koranyi approach regions in B:

$$D_\alpha(\zeta) = \{z \in B\colon |1 - \langle z, \zeta\rangle| < \alpha(1 - |z|)\}.$$

In this setting Theorem 9 becomes exactly Theorem 2.2 of [M], with $\alpha_0 = 1$ and $\alpha_1 = \sec\frac{\pi}{2N}$.

We finish with some brief remarks on examples. Consider the map $\phi\colon B_N \to B_N$ defined by

$$\phi(z_1, z_2, \ldots, z_N) = (N^{\frac{N}{2}} z_1 z_2 \cdots z_N, 0').$$

A direct computation ([MS, p. 904]) shows that for t small

$$\lambda_B^\alpha(\phi^{-1}(S(e_1, t)) \simeq t^{\alpha + \frac{(N-1)}{2} + 2},$$

where $e_1 = (1, 0') \in \partial B_N$.

Thus in this example C_ϕ is not bounded from $H^p(B_N)$ to $A^{p,\alpha}(B_N)$ for any $\alpha < \frac{N-3}{2}$ (notice that the spaces $A^{p,\alpha}(B_N)$ may be defined for any $\alpha > -1$), whereas Theorem 7 shows that C_ϕ must be bounded into $A^{p,\alpha}(B_N)$ for all $\alpha \geq N-2$. This leads naturally to the question of whether the weight $\alpha = N-2$ is optimal in Theorem 7. We believe it is, and in support of this we offer the following observation. For any $\phi\colon B_N \to B_N$ the same sort of slice integration argument used in the proof of Proposition 4 (specialized to the ball) shows that

$$\sigma_B\phi^{-1}S(\zeta, t) = O(t)$$

for $\zeta \in \partial B_N$ and $t > 0$ (again, we identify ϕ with its a.e. radial extension to ∂B_N). Moreover, the worst case situation $\sigma_B\phi^{-1}S(\zeta, t) \simeq t$ can occur: if γ is a non-constant inner function on B_N with $\gamma(0) = 0$ define ϕ on B_N by $\phi = (\gamma, 0')$. Then since γ is measure-preserving as a map from ∂B_N to $\partial\Delta$, $\sigma_B\phi^{-1}S(e_1, t) = t$. Unfortunately this example is not amenable to calculation of $\lambda_B^\alpha\phi^{-1}S(e_1, t)$ due to the bad oscillatory behavior of γ near points of ∂B_N. We do not know if $\sigma_B\phi^{-1}S(\zeta, t) \simeq t$ can occur, say, with a Lip 1 mapping ϕ; such an example would give $\lambda_B^\alpha\phi^{-1}S(\zeta, t) \simeq t^{\alpha+2}$, which in turn would show that the exponent $\alpha = N-2$ in Theorem 7 cannot be replaced by anything smaller.

The above inner function example can be used to show that Theorem 9 is optimal in the ball, as the relevant computations involve $\sigma_B\phi^{-1}$ rather than $\lambda_B^\alpha\phi^{-1}$. Specifically there exist $\phi\colon B_N \to B_N$ with $\phi(B_N) \subset D_{\alpha_1}(\zeta)$ ($\alpha_1 = \sec\frac{\pi}{2N}$) and C_ϕ bounded but not compact from $H^p(B_N)$ to $H^p(B_N)$, and for $\beta > \alpha_1$, there exist $\phi\colon B_N \to B_N$ with $\phi(B_N) \subset D_\beta(\zeta)$ yet C_ϕ not bounded from $H^p(B_N)$ to $H^p(B_N)$. See [M] for the details.

References

[A] M. Abate, *Common fixed points of commuting holomorphic maps*, Math. Ann. **283** (1989), 645–655.

[CW] J. A. Cima and W. R. Wogen, *Unbounded composition operators on* $H^2(B_2)$, Proc. Amer. Math. Soc. **99** (1987), 477–483.

[H] L. Hörmander, L^p *estimates for (pluri-)subharmonic functions*, Math. Scand. **20** (1967), 65–78.

[K] S. G. Krantz, *Function theory of several complex variables*, Wiley, New York, 1982.

[L1] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427–474.

[L2] ——, *Intrinsic distances and holomorphic retracts*, Complex Analysis and Applications, 1981, Bulgar. Acad. Sci., Sofia, Bulgaria, 1984, pp. 341–364.

[L3] ——, *A precise result on the boundary regularity of biholomorphic mappings*, Math. Z. **193** (1986), 559–579.

[L4] ——, *Holomorphic invariants, normal forms, and the moduli space of convex domains*, Ann. of Math. (2) **128** (1988), 43–78.

[LR] S. Y. Li and B. Russo, *On compactness of composition operators in Hardy spaces of several variables*, Proc. Amer. Math. Soc. **123** (1995), 161–171.

[M] B. D. MacCluer, *Compact composition operators on* $H^p(B_N)$, Michigan Math. J. **32** (1985), 237–248.

[MS] B. D. MacCluer and J. H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canad. J. Math. **38** (1986), 878–906.

[RW] H. L. Royden and P. M. Wong, *Carathéodory and Kobayashi metrics on convex domains*, preprint, 1983.

[R] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, Berlin, 1980.

[S] E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Princeton Univ. Press, Princeton, NJ, 1972.

[V] E. Vesentini, *Complex geodesics*, Compositio Math. **44** (1981), 375–394.

[W] W. R. Wogen, *The smooth mappings which preserve the Hardy space* $H^2(B_n)$, Oper. Theory: Adv. Appl. **35** (1988), 249–267.

Department of Mathematics and Computer Science, University of Richmond, Richmond, Virginia 23173

Current address: Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903

Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27599

Current address: Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

E-mail address: pmercer@math.purdue.edu