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DELAY DIFFERENTIAL INCLUSIONS WITH CONSTRAINTS

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ABSTRACT. In this paper we examine functional differential inclusions with memory and state constraints. For the case of time-independent state constraints, we show that the solution set is R_δ under Carathéodory conditions on the orientor field. For the case of time-dependent state constraints we prove two existence theorems. For this second case, the question of whether the solution set is R_δ remains open.

1. INTRODUCTION

Since the pioneering work of Aronszajn [1], several researchers have studied the regularity properties of the solution set of various differential equations and differential inclusions. Recall that a subset of a metric space is called an R_δ-set if it is the intersection of a decreasing sequence of nonempty, compact absolute retracts. In [1] it was proved that the solution set of the Cauchy problem $\dot{x}(t) = f(t, x(t))$, $x(0) = x_0$, with $t \in T = [0, r]$, $f(\cdot, \cdot)$ a bounded continuous vector field on $T \times \mathbb{R}^n$, is an R_δ-set, in particular then is acyclic. Due to the fixed point theorem of Eilenberg-Montgomery [4] for pseudo-acyclic operators, acyclicity is an important property since it can be used to establish the existence of periodic solutions for $\dot{x}(t) = f(t, x(t))$, provided that $f(t, x)$ is periodic in t. Aronszajn's results were extended to differential inclusions in $\mathbb{R}^n$ by Himmelberg-Van Vleck [9, 10] (autonomous systems) and DeBlasi-Myjak [3] (nonautonomous systems). In a recent paper [11], the authors proved the same regularity result for the solution set of differential inclusions on some $K \subseteq \mathbb{R}^n$ (here K is the set of state constraints).

The purpose of the present paper is to study delay differential inclusions with constraints. In case the constraint set K is time independent, we prove that the solution set is an R_δ-set and a periodic solution exists if the system is periodic. This way we generalize an earlier result of Haddad-Lasry [7]. When K is time-dependent, since the graph of K is not convex in general, we can only establish some existence results and pose as an open problem the question of whether the solution set is an R_δ-set.

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2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. We define $P_{f(c)}(X) = \{A \subseteq X\colon$ nonempty, bounded, closed (and convex)$\}$. A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf\{\|x-y\| : y \in F(\omega)\}$ is measurable.

Let Y, Z be Hausdorff topological spaces and $G : Y \to 2^Z\backslash\{\varnothing\}$. We say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c)), if for all $U \subseteq Z$ open, the set $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ (resp. $G^-(U) = \{y \in Y : G(y) \cap U \neq \varnothing\}$) is open in Y.

On $P_f(X)$ we can define the Hausdorff metric by

$$h(A, B) = \max\left[\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right].$$

The metric space $(P_f(X), h)$ is complete and a multifunction $G : X \to P_f(Y)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into $(P_f(Y), h)$.

If $K \in P_f(X)$ and $x \in K$, then the tangent cone to K at x is defined by

$$T_K(x) = \left\{ v \in X : \varliminf_{\lambda\downarrow 0} \frac{d(x+\lambda v, K)}{\lambda} = 0 \right\}.$$

This is a closed cone and is convex, if K is convex. If K is convex and $\operatorname{int} K \neq \varnothing$, then $\operatorname{int} T_K(x) \neq \varnothing$ also. The normal cone to K at x is defined by

$$N_K(x) = \{x^* \in X^* : (x^*, x) = \sigma(x^*, K) = \sup[(x^*, y) : y \in K]\}.$$

It is well known (see, for example, Aubin-Cellina [2]) that the normal cone is the negative polar cone of $T_K(x)$; i.e., $N_K(x) = T_K(x)^- = \{x^* \in X^* : (x^*, v) \leq 0$ for all $v \in T_K(x)\}$.

Recall that a set $A \subseteq X$ is contractible, if there exist a continuous $h : [0, 1] \times A \to A$ and $x_0 \in A$ such that $h(0, x) = x$ and $h(1, x) = x_0$. A set $C \subseteq X$ is said to be an absolute retract, if it can replace $\mathbb{R}$ in Tietze's theorem; i.e., for every metric space Y and closed $A \subseteq Y$, each continuous $f : A \to C$ has an extension $\hat{f} : Y \to C$. Evidently an absolute retract is contractible. To see this, let $Y = [0, 1] \times C$ and $A = \{0, 1\} \times C$ as a closed subset of Y, and consider on A the map $f(0, x) = x$ and $f(1, x) = x_0$ for $x \in C$. Hence an R_δ-set is the intersection of compact, contractible sets. In [12] Hyman showed that the converse is also true.

Definition [7]. Let X, Y be two metric spaces. A multifunction $\Gamma : X \to 2^Y\backslash\{\varnothing\}$ is said to be σ-selectionable, if there exists a sequence $\Gamma_n : X \to 2^Y\backslash\{\varnothing\}$ such that for each $n \geq 1$ $\Gamma_n(\cdot)$ is u.s.c., has a continuous selector and satisfies

(a) $\Gamma_{n+1}(x) \subseteq \Gamma_n(x)$ for all $n \geq 1$ and all $x \in X$;
(b) $\Gamma(x) = \bigcap_{n\geq 1} \Gamma_n(x)$ for all $x \in X$.

In order to prove the R_δ-property of the solution set of differential inclusions in [11], the authors proved the following lemma which is a generalization of Theorem 2.3 of Kisielewicz et al. [13] and will be used in the present paper.

Let $T = [0, r]$, X and Y be separable Banach spaces, and $K \subseteq X$ be closed. Recall that a function $f : T \times K \to Y$ is said to be Carathéodory if $t \to f(t, x)$ is measurable, $x \to f(t, x)$ is continuous and there exists $\varphi(\cdot) \in L^1(T)$ such that $|f(t, x)| \leq \varphi(t)$ a.e.

Lemma 1 [11]. *If $f : T \times K \to Y$ is a Carathéodory function, then for any $\varepsilon > 0$ there exists a jointly locally Lipschitz function*

$$f_\varepsilon : T \times K \to Y \text{ such that } \int_0^r \sup_{x \in K} \|f(t, x) - f_\varepsilon(t, x)\| \, dt \leq \varepsilon .$$

The following lemma is due to Rybinski [15] and is an extension of Michael's selection theorem.

Lemma 2 [15]. *If $G : T \times X \to P_{fc}(Y)$ is a measurable multifunction and for every $t \in T$, $G(t, \cdot)$ is l.s.c., then there exists $g : T \times X \to Y$ such that $t \to g(t, x)$ is measurable, $x \to g(t, x)$ is continuous and for all $(t, x) \in T \times X$ we have $g(t, x) \in G(t, x)$.*

The following fixed point result is due to Haddad-Lasry [7]:

Lemma 3 [7]. *Let $H \subseteq X$ be nonempty, convex, and compact and assume that Γ is a σ-selectionable multifunction from H into X such that $\Gamma(x) \subseteq H$ for all $x \in H$. Then $\Gamma(\cdot)$ has a fixed point in H (i.e., there exists $x \in H$ such that $x \in \Gamma(x)$).*

By homology we mean Čech homology with rational coefficients and call a compact metric space Z acyclic if it has the same homology as a one-point space. Let V, W be metric spaces and $T : V \to 2^W \backslash \{\varnothing\}$. We say that $T(\cdot)$ is acyclic if $T(v)$ is compact and acyclic for every $v \in V$. Then $F : V \to 2^V \backslash \{\varnothing\}$ is said to be pseudo-acyclic if $F = roT$, where $T(\cdot)$ is u.s.c. and acyclic and $r : W \to V$ is a single-valued, continuous map. The following basic fixed point theorem for pseudo-acyclic maps is due to Eilenberg-Montgomery [4].

Lemma 4 [4]. *Let M be an acyclic absolute neighborhood retract (ANR), N a compact metric space, $r : N \to M$ a continuous single-valued map and $T : M \to 2^N \backslash \{\varnothing\}$ an u.s.c. acyclic multifunction. Then $roT : M \to 2^M \backslash \{\varnothing\}$ has a fixed point (i.e., there exist $x \in M$ such that $x \in (roT)(x)$).*

3. Inclusions with constant constraints

For a given $\tau > 0$ and $\omega > 0$, let $J = [-\tau, 0]$, $T = [0, \omega]$, and $C = C(J, \mathbb{R}^n)$. For a nonempty compact and convex $K \subseteq \mathbb{R}^n$, define $\mathscr{K} = \{\varphi \in C : \varphi(\xi) \in K \text{ on } J\}$. Assume that $F : T \times \mathscr{K} \to 2^{\mathbb{R}^n} \backslash \{\varnothing\}$ and $\varphi \in \mathscr{K}$ are given. We consider the following functional differential inclusion with memory:

$$(1) \qquad \left\{ \begin{array}{c} \dot{x}(t) = F(t, x_t) \quad \text{a.e. on } T \\ x(\xi) = \varphi(\xi), \quad \xi \in J, \qquad x(t) \in K, \quad t \in T. \end{array} \right\}$$

Here $x_t(\cdot) \in C$ is defined by $x_t(\xi) = x(t + \xi)$. Denote by $S(\varphi)$ the set of all solutions of (1) with initial function $\varphi(\cdot)$ and for $t \in T$ define $S_t : \mathscr{K} \to 2^{\mathscr{K}} \backslash \{\varnothing\}$ by $S_t(\varphi) = \{x(t + \cdot) \in \mathscr{K} : x \in S(\varphi)\}$. We will show that under some reasonable conditions, $S(\varphi)$ is an R_δ-set for every $\varphi \in \mathscr{K}$ and (1) possesses an ω-periodic solution if $F(t, x)$ is ω-periodic in its first variable.

The basic hypotheses on the data of (1) are the following:
H_1: $F : T \times \mathscr{K} \to P_{fc}(\mathbb{R}^n)$ is a multifunction such that

(1) $t \to F(t, \psi)$ is measurable,
(2) $\psi \to F(t, \psi)$ is u.s.c.,
(3) $\sup\{\|y\| : y \in F(t, \psi)\} \leq M$ for all $(t, \psi) \in T \times \mathscr{K}$.

H_2: $K \subseteq \mathbb{R}^n$ is nonempty, compact, and convex.
H_3: $F(t, \psi) \cap T_K(\psi(0)) \neq \varnothing$ for all $(t, \psi) \in T \times \mathscr{K}$.

Theorem 1. *If hypotheses $H_1 \to H_3$ hold, then*
(i) *for all $\varphi \in \mathscr{K}$, $S(\varphi)$ is a nonempty R_δ-set in $C(\widehat{T}, \mathbb{R}^n)$ with $\widehat{T} = [-\tau, \omega]$,*
(ii) *$\varphi \to S(\varphi)$ is σ-selectionable on $\mathscr{K}$,*
(iii) *(1) has an ω-periodic solution, if $F(\cdot, x)$ is ω-periodic.*

Proof. Without any loss of generality, we may assume that $\operatorname{int} K \neq \varnothing$. Indeed, if this is not the case, let $X_0 = \operatorname{span} K$. This is a subspace of $\mathbb{R}^n$ and clearly K has nonempty interior in X_0. Furthermore, it is easy to see that $T_K(x) \subseteq X_0$ for all $x \in K$ and the orientor field $F(t, \psi) \cap X_0$ satisfies H_3. Hence we can consider the following problem equivalent to (1):

$$(1)' \qquad \left\{ \begin{array}{c} \dot{x}(t) \in F(t, x_t) \cap X_0 \quad \text{a.e. on } T \\ x(\xi) = \varphi(\xi), \quad \xi \in J, \qquad x(t) \in K, \quad t \in T. \end{array} \right\}$$

Finally through a translation if necessary, we can always have $0 \in \operatorname{int} K$. Thus there exists $\delta > 0$ such that for all $x^* \in \mathbb{R}^n$

$$(2) \qquad \delta\|x^*\| \leq \sigma(x^*, K) = \sup\{(x^*, x) : x \in K\}.$$

Invoking Lemma 3 of DeBlasi-Myjak [3], we get multifunctions $F_0, G_n : T \times \mathscr{K} \to P_{fc}(\mathbb{R}^n)$ such that: (1) $F_0(t, \psi) \subseteq F(t, \psi)$ for all $(t, \psi) \in T \times \mathscr{K}$ and if $\Delta \subseteq T$ is measurable and $\eta : \Delta \to \mathscr{K}$, $y : \Delta \to \mathbb{R}^n$ are measurable and $y(t) \in F(t, \eta(t))$ a.e., then $y(t) \in F_0(t, \eta(t))$ a.e. on Δ; (2) $G_n(\cdot, \psi)$ is measurable and $G_n(t, \cdot)$ is h-continuous; (3) $F_0(t, \psi) \subseteq G_{n+1}(t, \psi) \subseteq G_n(t, \psi)$ for all $n \geq 1$ and all $(t, \psi) \in T \times \mathscr{K}$; (4) $h(G_n(t, \psi), F_0(t, \psi)) \to 0$ as $n \to \infty$ for all $(t, \psi) \in T \times \mathscr{K}$; (5) $|G_n(t, \psi)| = \sup\{\|z\| : z \in G_n(t, \psi)\} \leq M_1$ for some $M_1 > 0$ and all $(t, \psi) \in T \times \mathscr{K}$. From property (1) above, it is clear that problem (1) is equivalent to

$$(3) \qquad \left\{ \begin{array}{c} \dot{x}(t) \in F_0(t, x_t) \quad \text{a.e. on } T \\ x(\xi) = \varphi(\xi), \quad \xi \in J, \qquad x(t) \in K, \quad t \in T. \end{array} \right\}$$

In general, we cannot guarantee that the tangential hypothesis H_3 holds for $F_0(t, x)$. Therefore instead of (3), we consider the following approximating problem:

$$(4)_n \qquad \left\{ \begin{array}{c} \dot{x}(t) \in G_n(t, x_t) \quad \text{a.e. on } T \\ x(\xi) = \varphi(\xi), \quad \xi \in J, \qquad x(t) \in K, \quad t \in T. \end{array} \right\}$$

We claim that

$$(5) \qquad G_n(t, \psi) \cap T_K(\psi(0)) \neq \varnothing \quad \text{for all } (t, \psi) \in T \times \mathscr{K}.$$

To this end, given $\varepsilon > 0$ by the Scorza-Dragoni theorem, there exist closed $T_0 \subseteq T$ with $\lambda(T \backslash T_0) < \varepsilon$ (here $\lambda(\cdot)$ is the Lebesgue measure on T) such that

$G_n|_{T_0 \times \mathscr{K}}$ is h-continuous. Since almost all points in T_0 are points of density, there is a closed $T_\varepsilon \subseteq T_0$ with $\lambda(T_0 \backslash T_\varepsilon) < \varepsilon$ such that T_ε contains only points of density of T_0. Then since $\varepsilon > 0$ is arbitrary, it is enough to prove (5) for all $(t, \psi) \in T_\varepsilon \times \mathscr{K}$.

Fix $t_0 \in T_\varepsilon$. Note that the multifunction $t \to F(t, \psi) \cap T_K(\psi(0))$ is measurable (cf. Himmelberg [8]). So by the Kuratowski-Ryll Nardzewski selection theorem (see, for example, Himmelberg [8]), we can find $u : T \to \mathbb{R}^n$ measurable such that $u(t) \in F(t, \psi) \cap T_K(\psi(0))$ a.e. Thus $u(t) \in F_0(t, \psi) \cap T_K(\psi(0))$ and so a fortiori $u(t) \in F(t, \psi) \cap T_K(\psi(0))$ for all $t \in T \backslash N(\psi)$, for some Lebesgue-null set $N(\psi) \subseteq T$. Then there exist $t_m \in T_0 \backslash N(\psi)$ such that $t_m \to t_0$ and let $y_m = u(t_m) \in G_n(t_m, \psi) \cap T_K(\psi(0))$, $m \geq 1$. Upon taking the limit of a convergent subsequence of $\{y_n\}_{m \geq 1}$, since $G_n|_{T_0 \times \mathscr{K}}$ is h-continuous and $T_K(\psi(0))$ is closed, we get for some $y_0 \in \mathbb{R}^n$ that $y_0 \in G_n(t_0, \psi) \cap T_K(\psi(0)) \neq \varnothing$. So we have proved (5).

Next for $0 < \varepsilon \leq \delta$, define $G_n^\varepsilon(t, \psi) = G_n(t, \psi) + B_\varepsilon$, where $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| \leq \varepsilon\}$. It is obvious that $G_n^\varepsilon(t, \psi) \cap \operatorname{int} T_K(\psi(0)) \neq \varnothing$. Since $G_n(t, \psi)$ is measurable in t and h-continuous in ψ, then $(t, \psi) \to G_n^\varepsilon(t, \psi)$ is measurable (cf. Papageorgiou [14]) and so $(t, \psi) \to G_n^{\varepsilon/3}(t, \psi) \cap T_K(\psi(0))$ is measurable. Since $\psi \to \operatorname{int} T_K(\psi(0))$ has an open graph (see, for example, Aubin-Cellina [2]), lemma β of Flytzanis-Papageorgiou [5] implies that $\psi \to G_n^\varepsilon(t, \psi) \cap \operatorname{int} T_K(\psi(0))$ is l.s.c., therefore $\psi \to \overline{G_n^{\varepsilon/3}(t, \psi) \cap \operatorname{int} T_K(\psi(0))} = G_n^{\varepsilon/3}(t, \psi) \cap T_K(\psi(0))$ is l.s.c. Hence according to Lemma 2, we can find $g_n^{1,\varepsilon} : T \times \mathscr{K} \to \mathbb{R}^n$ a function measurable in t, continuous in ψ such that

$$g_n^{1,\varepsilon}(t, \psi) \in G_n^{\varepsilon/3}(t, \psi) \cap T_K(\psi(0)) \quad \text{for all } (t, \psi) \in T \times \mathscr{K}. \tag{6}$$

Invoking Lemma 1, for any $\theta > 0$ we can find $g_n^{2,\varepsilon} : T \times \mathscr{K} \to \mathbb{R}^n$ which is locally Lipschitz in (t, x) and such that

$$\int_0^\omega \sup_{\psi \in \mathscr{K}} \|g_n^{1,\varepsilon}(t, \psi) - g_n^{2,\varepsilon}(t, \psi)\| \, dt < \theta.$$

It is clear that for each fixed $n \geq 1$, by choosing $\theta > 0$ sufficiently small, we can get measurable $A_n \subseteq T$ with $\lambda(A_n) < \frac{1}{n^2}$ such that

$$\|g_n^{1,\varepsilon}(t, \psi) - g_n^{2,\varepsilon}(t, \psi)\| < \frac{\varepsilon\delta}{3} \tag{7}$$

for all $(t, \psi) \in (T \backslash A_n) \times \mathscr{K}$. Set $D_n = \bigcup_{k=n}^\infty A_k$, $\varepsilon = \frac{1}{n}$ and define $G_n^*(t, \psi) = G_n(t, \psi) + B_{\frac{1}{n}} + \chi_{D_n}(t) B_{\beta/\delta}$, where $\beta = \hat{m} + 1$ with $\hat{m} = \sup\{\|g_n^{2,\frac{1}{n}}(t, \psi)\| : (t, \psi) \in T \times \mathscr{K}\}$.

It is clear that $G_n^*(t, \psi) \in P_{fc}(\mathbb{R}^n)$, $t \to G_n^*(t, \psi)$ is measurable and $\psi \to G_n^*(t, \psi)$ is h-continuous. We consider

$$(8)_n \qquad \left\{ \begin{array}{ll} \dot{x}(t) \in G_n^*(t, x_t) & \text{a.e. on } T \\ x(\xi) = \varphi(\xi), \quad \xi \in J, & x(t) \in K, \quad t \in T. \end{array} \right\}$$

Denote the solution set of $(8)_n$ by $S_n^*(\varphi)$. From Haddad [6], we know that $S_n^* \subseteq C(\widehat{T}, \mathbb{R}^n)$ is nonempty and compact $(\widehat{T} = [-\tau, \omega])$. Also since $\lambda(D_n) \leq \sum_{k=n}^\infty \frac{1}{k^2} \to 0$ and for all $(t, \psi) \in T \times \mathscr{K}$ $h(G_n(t, \psi), F_0(t, \psi)) \to 0$ as $n \to \infty$, we have that $h(G_n^*(t, \psi), F_0(t, \psi)) \to 0$ in measure and thus by passing to a subsequence if necessary $h(G_n^*(t, \psi), F_0(t, \psi)) \to 0$ a.e. Consequently, we have $S_{n+1}^*(\varphi) \subseteq S_n^*(\varphi)$, $n \geq 1$, and

$$S(\varphi) = \bigcap_{n \geq 1} S_n^*(\varphi). \tag{9}$$

Now as in Yorke [16, Theorem 1.3] and in Himmelberg-Van Vleck [10, Proposition 3], we will show that $S_n^*(\varphi)$ is contractible. Also we will show that $S_n^*(\cdot)$ admits a continuous selector and so we will have proved the first two conclusions of the theorem.

For each $n \geq 1$, let $g_n^*(t, \psi) = g_n^{2, \frac{1}{n}}(t, \psi) - \frac{1}{3n}\psi(0) - \frac{\beta}{\delta}\chi_{D_n}(t)\psi(0)$, where $g_n^{2, \frac{1}{n}}(t, \psi)$ is from (7). Consider the following problem; here $0 \leq s < \omega$ and $\psi \in \mathscr{K}$:

$$\left\{ \begin{array}{ll} \dot{x}(t) = g_n^*(t, x_t) \quad \text{a.e. on } [s, \omega] & \\ x(s + \xi) = \varphi(\xi), \quad \xi \in J, & x(t) \in K, \quad t \in [s, \omega]. \end{array} \right\} \tag{10_n}$$

We claim that $g_n^*(t, \psi) \in T_K(\psi(0))$ for all $\psi \in \mathscr{K}$. Indeed let $x^* \in N_K(\psi(0))$ and first assume that $t \notin D_n$. Then

$$\begin{aligned} (x^*, g_n^*(t, \psi)) &= (x^*, g_n^{2, \frac{1}{n}}(t, \psi)) - \frac{1}{3n}(x^*, \psi(0)) \\ &= (x^*, g_n^{2, \frac{1}{n}}(t, \psi) - g_n^{1, \frac{1}{n}}(t, \psi)) \\ &\quad + (x^*, g^{1, \frac{1}{n}}(t, \psi)) - \frac{1}{3n}(x^*, \psi(0)) \\ &\leq \frac{\delta}{3n}\|x^*\| - \frac{\delta}{3n}\|x^*\| = 0 \end{aligned}$$

(recall that $x^* \in N_K(\psi(0))$ means that $(x^*, \psi(0)) = \sigma(x^*, K)$, hence $(x^*, \psi(0)) \geq \delta\|x^*\|$ from (2)). Thus $g_n^*(t, \psi) \in T_K(\psi(0))$ when $t \notin D_n$.

Next let $t \in D_n$. Then for $x^* \in N_K(\psi(0))$ we have

$$\begin{aligned} (x^*, g_n^*(t, \psi)) &= (x^*, g_n^{2, \frac{1}{n}}(t, \psi)) - \frac{1}{3n}(x^*, \psi(0)) - \frac{\beta}{\delta}(x^*, \psi(0)) \\ &\leq \|x^*\|\left(\hat{m} - \frac{\delta}{3n} - \beta\right) \leq 0. \end{aligned}$$

Therefore, for all $(t, \psi) \in T \times \mathscr{K}$, we have $g_n^*(t, \psi) \in T_K(\psi(0))$. It is clear that $t \to g_n^*(t, \psi)$ is measurable, while $\psi \to g_n^*(t, \psi)$ is locally Lipschitz. It is then easy to check that $(10)_n$ has a unique solution $u_n(t; s, \psi)$ for any $s \in [0, \omega)$ and $\psi \in \mathscr{K}$. Furthermore $(s, \psi) \to u_n(\cdot; s, \psi)$ is continuous. Also note that $g_n^*(t, \psi) \in G_n^*(t, \psi)$ for all $(t, \psi) \in T \times \mathscr{K}$, hence $u_n(\cdot; 0, \psi) \in S_n^*(\psi)$ and so we have shown that $S_n^*(\cdot)$ admits a continuous selector. Next we define a homotopy $h : [0, 1] \times S_n^*(\varphi) \to S_n^*(\varphi)$ as follows

$$h(\lambda, x)(t) = \begin{cases} \varphi(t) & \text{if } t \in [-\tau, 0], \\ x(t) & \text{if } t \in [0, \lambda\omega], \\ u(t; \lambda\omega, x_{\lambda\omega}) & \text{if } t \in [\lambda\omega, \omega]. \end{cases} \tag{11}$$

This shows that $S_n^*(\varphi)$ is contractible.

Finally, we assume that $F(\cdot, \psi)$ is ω-periodic and then prove that (1) has an ω-periodic solution. To this end, let $\widehat{\mathscr{K}} = \{\varphi \in AC([-\tau, 0], \mathbb{R}^n) : \|\dot{\varphi}(t)\| \leq M$ a.e.$\}$ (here by $AC([-\tau, 0], \mathbb{R}^n)$ we denote the space of absolutely continuous functions on $[-\tau, 0]$ into $\mathbb{R}^n$). Let $N = \overline{\bigcup_{\varphi \in \widehat{\mathscr{K}}} S(\varphi)}$. Then N is compact in $C(\widehat{T}, \mathbb{R}^n)$, where $\widehat{T} = [-\tau, \omega]$ (Arzela-Ascoli theorem). Recalling that $S : \widehat{\mathscr{K}} \to 2^{C(\widehat{T}, \mathbb{R}^n)} \backslash \{\varnothing\}$ is u.s.c. with compact acyclic values (in fact, R_δ-values) and noting that $S_\omega : \widehat{\mathscr{K}} \to \widehat{\mathscr{K}}$, we can apply Lemma 4 on the pseudo-acyclic operator S_ω and thus get $\varphi \in S_\omega(\varphi)$. Hence there exists $x(\cdot) \in S(\varphi)$ such that $x(\omega + \xi) = x(\xi)$ for every $\xi \in J = [-\tau, 0]$; i.e., this is the periodic trajectory. Q.E.D.

Remarks. (1) Part (iii) of this theorem extends Theorem B.II.1 of Haddad-Lasry [7], who assume that the orientor field $F(t, \psi)$ is jointly u.s.c.

(2) The existence of periodic solutions may also be proved via Lemma 3, but the approach using Lemma 4 that we follow here is much more straightforward.

4. Inclusions with time-varying constraints

In this section we assume that K depends on t. So let $K : T = [0, \omega] \to 2^{\mathbb{R}^n} \backslash \{\varnothing\}$, $\widehat{K} = \operatorname{gr} K = \{(t, x) \in T \times \mathbb{R}^n : x \in K(t)\}$, $\mathscr{K}(t) = \{\varphi \in C : \varphi(0) \in K(t)\}$ and $\widehat{\mathscr{K}} = \operatorname{gr} \mathscr{K} = \{(t, \varphi) \in T \times C : \varphi \in \mathscr{K}(t)\}$. Let $F : \widehat{\mathscr{K}} \to 2^{\mathbb{R}^n} \backslash \{\varnothing\}$ and $(t_0, \varphi) \in \widehat{\mathscr{K}}$ with $0 \leq t_0 < \omega$ be given. We consider the following delay differential inclusion with time varying constraints:

$$
\left\{ \begin{array}{c} \dot{x}(t) \in F(t, x_t) \quad \text{a.e. on } [t_0, \omega] \\ x(\xi) = \varphi(\xi - t_0), \quad \xi \in [t_0 - \tau, t_0], \qquad x(t) \in K(t), \quad t \in [t_0, \omega]. \end{array} \right\} \tag{12}
$$

In what follows by $DK(t, x)$ we denote the contingent derivative of $K(\cdot)$ at $(t, x) \in \operatorname{gr} K$, i.e., $v \in DK(t, x)(u)$ if and only if $(u, v) \in T_{\widehat{K}}(t, x)$ (cf. Aubin-Cellina [2]).

Theorem 2. *Assume that there exist constants M, $M' > 0$ such that*

(1) *$K(t) \subseteq B_M$ for all $t \in T$ and $t \to K(t)$ is u.s.c. with closed values,*

(2) *$F(t, \varphi) \subseteq B_M$ on $\widehat{\mathscr{K}}$ and $F(\cdot, \cdot)$ is jointly u.s.c. with closed and convex values.*

Then problem (12) *above admits a solution for any $(t_0, \varphi) \in \widehat{\mathscr{K}}$ with $t_0 \in [0, \omega)$ if and only if*

$$
F(t, \psi) \cap DK(t, \psi(0))(1) \neq \varnothing \tag{13}
$$

for all $(t, \psi) \in \widehat{\mathscr{K}}$.

Proof. Sufficiency: We follow a well-known procedure; i.e., we transform (12) into a system with constant constraints. To this end, let $\widehat{C} = C(J, \mathbb{R}^{1+n})$ and $\mathscr{M} = \{\psi^* \in \widehat{C} : \psi^*(0) \in \widehat{K}\}$. Define $G : \mathscr{M} \to 2^{\mathbb{R}^{1+n}} \backslash \{\varnothing\}$ by

$$
G(\psi^*) = (1, F(t(0), \psi)) \tag{14}
$$

for $\psi^* = (t(\cdot), \psi(\cdot)) \in \mathscr{M}$, with $t(\cdot) \in C(J, \mathbb{R})$ and $\psi(\cdot) \in C(J, \mathbb{R}^n)$. Note that $G(\cdot)$ is well defined, since $\psi^* \in \mathscr{M}$ implies $(t(0), \psi(0)) \in \widehat{K}$, hence $F(t(0), \psi)$ is defined. Let $\varphi^*(\xi) = (t_0 + \xi, \varphi(\xi))$ for $\xi \in J$. Then $\varphi^* \in \mathscr{M}$. Let $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$. Consider the following autonomous delay differential inclusion with constant constraints:

$$(15)\quad \left\{\begin{array}{c} \dot{y}(\sigma) \in G(y_\sigma) \quad \text{on } [t_0, \omega] \\ y(\xi) = \varphi^*(\xi - t_0), \quad \xi \in [t_0 - \tau, t_0], \qquad y(\sigma) \in \widehat{K}, \quad \sigma \in [t_0, \omega]. \end{array}\right\}$$

It is easy to check that (12) has a solution for φ if and only if (15) has a solution for φ^*, where $\varphi^*(\xi) = (t_0 + \xi, \varphi(\xi))$, $\xi \in J$. Hence we only need to show that (15) has a solution.

It is clear that $G(\cdot)$ is u.s.c. with compact, convex values. Furthermore, since (13) is equivalent to $(1, F(t, \psi)) \cap T_{\widehat{K}}(t, \psi(0)) \neq \varnothing$, we have for any $\psi^* \in (t, \psi) \in \mathscr{M}$,

$$(16)\qquad G(\psi^*) \cap T_{\widehat{K}}(\psi^*(0)) \neq \varnothing.$$

Consequently, Theorem II-1 of Haddad [6] applies and yields a solution for (15), hence for (12) also.

Necessity: Without any loss of generality, we only prove that (13) is true for $t = t_0$. Thus assume that $(t_0, \varphi) \in \widehat{\mathscr{H}}$ is given and the corresponding differential inclusion (12) has a solution. Since $x(\cdot)$ is continuous, $x_{t_0}(\cdot) = \varphi(\cdot)$, $F(\cdot, \cdot)$ is u.s.c. at (t_0, φ) and $F(t_0, \varphi)$ is compact and convex, for any $\varepsilon > 0$ we have that

$$(17)\qquad \frac{x(t) - x(t_0)}{t - t_0} = \frac{1}{t - t_0} \int_{t_0}^{t} \dot{x}(\eta)\, d\eta \in F(t_0, \varphi) + B_\varepsilon$$

with $t > t_0$ sufficiently close to t_0. Define

$$V = \left\{ y \in \mathbb{R}^n : \frac{x(t_n) - x(t_0)}{t_n - t_0} \to y \text{ for some } t_n \downarrow t_0 \right\}.$$

It is easy to verify that $(1, V) \subseteq T_{\widehat{K}}(t_0, \varphi(0))$, hence (17) implies

$$(1, V) \subseteq (1, F(t_0, \varphi)) \cap T_{\widehat{K}}(t_0, \varphi(0)).$$

Since $x(\cdot)$ is Lipschitzian on $[t_0, \omega]$, we have that $V \neq \varnothing$ and consequently (13) is true with $t = t_0$. Q.E.D.

In Theorem 2, we established the existence of a solution for (12) such that $(t, x) \in \widehat{K}$. This suggests that in order to find solutions satisfying a time-varying constraint, maybe $F(\cdot, \cdot)$ need not be defined on all of $\widehat{\mathscr{H}}$, but only on a proper subset of it. In the next theorem we show that indeed this is possible. So to simplify things, assume $t_0 = 0$. Given an initial function $\varphi \in C$, with $\varphi(0) \in K(0)$, we first extend K on $[-\tau, \omega]$ by setting $K(\xi) = \overline{\text{conv}}\{K(0), \varphi(\xi)\}$ for $\xi \in J$. Then it is clear that on $J = [-\tau, 0]$, $K(\cdot)$ is h-continuous. For $t \in T$ define $\mathscr{H}_0(t) = \{\psi \in C : \psi(\xi) \in K(t + \xi) \text{ for } \xi \in J\}$ and let $\widehat{\mathscr{H}_0} = \text{gr}\,\mathscr{H}_0 = \{(t, \psi) \in T \times C : \psi \in \mathscr{H}_0(t)\}$.

Theorem 3. *Assume that $F : \widehat{\mathscr{H}_0} \to 2^{\mathbb{R}^n} \backslash \{\varnothing\}$ is a jointly u.s.c. multifunction with compact convex values and $F(t, \psi) \subseteq B_M$ for all $(t, \psi) \in \widehat{\mathscr{H}_0}$. Also assume that $K : T \to 2^{\mathbb{R}^n} \backslash \{\varnothing\}$ is h-continuous with compact convex values. Then given $\varphi \in C$ and if for all $(t, \psi) \in \widehat{\mathscr{H}_0}$, $F(t, \psi) \cap DK(t, \psi(0))(1) \neq \varnothing$, problem* (12) *admits a solution.*

Proof. For $(t, \xi) \in T \times J$, let $P(t, \xi)$ be the metric projection from $\mathbb{R}^n$ to $K(t+\xi)$. Since $K(\cdot)$ has compact convex values, this map is single valued and nonexpansive. Also define the operator $\mathscr{P}(t) : \mathscr{H}(t) \to \mathscr{H}_0(t)$ by $(\mathscr{P}(t)\psi)(\xi) = P(t, \xi)\psi(\xi)$. We have $\mathscr{P}(t)\psi \in \mathscr{H}_0(t)$ (recall that $K|_{[-\tau, 0]}$ is h-continuous). Define $G : \widehat{H} \to 2^{\mathbb{R}^n}\setminus\{\varnothing\}$ by $G(t, \psi) = F(t, \mathscr{P}(t)\psi)$ (this is well defined since $\mathscr{P}(t)\psi \in \mathscr{H}_0(t)$ for every $(t, \psi) \in T \times C$). It is easy to see that $G(\cdot, \cdot)$ is u.s.c. and has compact and convex values. Moreover, for $(t, \psi) \in \widehat{\mathscr{H}}$, since $(\mathscr{P}(t)\psi)(0) = P(t, 0)\psi(0) = \psi(0)$, we get from the tangential hypothesis that

$$G(t, \psi) \cap DK(t, \psi(0)) = F(t, \mathscr{P}(t)\psi) \cap DK(t, (\mathscr{P}(t)\psi)(0)) \neq \varnothing .$$

Therefore we can apply Theorem 2 to obtain a solution $x(\cdot)$ for

$$\left\{\begin{array}{c} \dot{x}(t) \in G(t, x_t) \quad \text{a.e. on } T = [0, \omega] \\ x(\xi) = \varphi(\xi), \quad \xi \in J = [-\tau, 0], \qquad x(t) \in K(t), \quad t \in T. \end{array}\right\}$$

In particular, $\dot{x}(t) \in F(t, \mathscr{P}(t)x_t)$ a.e., $\mathscr{P}(t)x_t = P(t, \xi)x(t+\xi)$ and $x(t+\xi) \in K(t+\xi)$ for all $(t, \xi) \in T \times J$. Hence $\mathscr{P}(t)x_t = x_t$ for $t \in T$ and consequently, $x(\cdot)$ solves (12). Q.E.D.

Remark. In general, $\widehat{K}$ is not convex, hence the arguments in the proof of Theorem 1 are not applicable when time-dependent constraints are present. So the question of whether the solution set of (12) is an R_δ-set remains open.

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