PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 7, July 1995

ON THE TORSION PART OF $\mathbb{C}^{[n]}$ WITH RESPECT TO THE ACTION OF A DERIVATION

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(Communicated by Hal L. Smith)

ABSTRACT. We show that the torsion part of $\mathbb{C}^{[n]}$ with respect to the action of a derivation is algebraically closed in $\mathbb{C}^{[n]}$ if the flow associated with the derivation is analytic on $\mathbb{C} \times \mathbb{C}^n$. We also present a connection between this result and Keller's Jacobian conjecture.

1. INTRODUCTION

Consider the initial value problem

$$\dot{y}\ (\equiv \frac{dy}{dt}) = \mathbf{V}(y), \qquad y(0) = x \in \mathbb{F}^n \tag{1.1}$$

where $\mathbf{V}$ is a polynomial vector field on $\mathbb{F}^n$ ($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Let $\phi: \Omega \to \mathbb{F}^n$ be the (local) flow associated with (1.1) where Ω, an open subset of $\mathbb{R} \times \mathbb{F}^n$, is the maximal domain of ϕ. It follows that ϕ extends to a function holomorphic on some open subset of $\mathbb{C} \times \mathbb{C}^n$. With $\mathbf{V}$ we associate the derivation $D = \mathbf{V}(X_1, \dots, X_n) \cdot \nabla$ of the ring of polynomials $\mathbb{C}^{[n]}$ in $X_1, \dots, X_n$ over $\mathbb{C}$. The *torsion part* $T(D)$ of $\mathbb{C}^{[n]}$ with respect to D is the set of polynomials a in $\mathbb{C}^{[n]}$ for which there exists a nonzero polynomial $q(D)$ in $\mathbb{C}[D]$ such that $q(D)a = 0$.

The main result in this paper is that the torsion part of $\mathbb{C}^{[n]}$ with respect to a derivation is algebraically closed in $\mathbb{C}^{[n]}$ if the flow associated with the derivation is analytic on $\mathbb{C} \times \mathbb{C}^n$.

We say that ϕ is a *polynomial flow* if each t-advance map ϕ^t is polynomial on its natural domain. Coomes and Zurkowski [8] show that ϕ is a polynomial flow if and only if $T(D) = \mathbb{C}^{[n]}$. (See [3,5,6,7,10,11,12,13,14,15,17] for other results about polynomial flows.) Since the torsion part $T(D)$ can play an important role in determining whether a vector field has a polynomial flow, we wish to gain a deeper understanding of the properties of $T(D)$. In addition, Connell and Drost [4] give a concise proof of the fact that the kernel of D is algebraically closed in $\mathbb{C}^{[n]}$. Since $T(D)$ is a generalization of the kernel of D, it is natural to consider questions about its algebraic closure in $\mathbb{C}^{[n]}$.

This paper is organized as follows: In section 2 we show that $T(D)$ is the set of polynomials in $\mathbb{C}^{[n]}$ which, when composed with ϕ, are polynomial in

Received by the editors November 16, 1993.
1991 *Mathematics Subject Classification*. Primary 34A20; Secondary 30D15, 12H05.

the initial condition (a notion made precise in that section). Then, in section 3 we show the set $\mathbb{C}[z]$ of polynomials in z over $\mathbb{C}$ is algebraically closed in the set of entire functions and we give a generalization. Our main result follows quickly in section 4. Finally, in section 5 we present a connection between our results and Keller's Jacobian conjecture.

2. A CHARACTERIZATION OF $T(D)$

Throughout this section, let ϕ be the flow associated with (1.1), let D be the derivation associated with $\mathbf{V}$, and let $T(D)$ be the corresponding torsion part of $\mathbb{C}^{[n]}$. For b in $\mathbb{C}^{[n]}$, denote the evaluation of b at x by $b|_x$. The following definition is motivated by that of polynomial flows.

Definition. Let A be any set, and let $g: A \times \mathbb{F}^k \to \mathbb{F}$. We say that $g(t, x)$ is *polynomial in its last* k *variables* if there is an integer d such that for each a in A, the function $f_a(x) = g(a, x)$ is polynomial of degree less than d. In this case, we say that the *degree of* g *in its last* k *variables* is the least upper bound of all such integers d.

Lemma 2.1. *The torsion part* $T(D)$ *of* $\mathbb{C}^{[n]}$ *is the set of all polynomials* a *in* $\mathbb{C}^{[n]}$ *such that the composition* $a \circ \phi = a|_\phi$ *is polynomial in the initial condition* x.

Proof. Let a be in $T(D)$, and suppose that $p(D)a = 0$ where p is a nonzero element of $\mathbb{C}[D]$. Making use of identities described by Coomes and Zurkowski [8], since

$$(p(D)a)|_\phi = p(\partial_t)(a|_\phi) = 0,$$

it follows that there are integers m and ℓ_i, functions $b_{ij}(x)$, and complex numbers λ_i such that

$$a|_\phi = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i} b_{ij}(x) t^j e^{\lambda_i t}/j!.$$

Furthermore, there exist polynomials $q_{ij}(D)$ in $\mathbb{C}[D]$ such that

$$b_{ij} = q_{ij}(D)a.$$

Hence $a|_\phi$ is polynomial in its last n variables.

Suppose now that $a|_\phi$ is polynomial in its last n variables. Then

$$a|_\phi = \sum_{|r| \le d} c_r(t) x^r$$

on the natural domain Ω of ϕ. Notice that the functions $c_r(t)$ are analytic near $t = 0$. Since

$$D^k a|_x = \sum_{|r| \le d} c_r^{(k)}(0) x^r,$$

it follows that for some integer $N > 0$ and some set of scalars $\alpha_0, \dots, \alpha_{N-1}$ we have

$$D^N a - \sum_{i=0}^{N-1} \alpha_i D^i a = 0.$$

That is, a is in $T(D)$. $\square$

3. ALGEBRAIC CLOSURE AND POLYNOMIALS

In this section, we first show in Lemma 3.1 that the set $\mathbb{C}[z]$ of polynomials in z over $\mathbb{C}$ is algebraically closed in the set of entire functions. This is a special case of Proposition D.1 of Bass [1], but the lemma we prove includes a degree estimate needed in Lemma 3.2. In that lemma we extend Lemma 3.1 to functions of more than one variable which are polynomial in some variables.

Lemma 3.1. *Let $f:\mathbb{C} \to \mathbb{C}$ be entire, and suppose that there exists an integer $m > 0$ and polynomials $a_0, \ldots, a_m$ in $\mathbb{C}[z]$ with $a_m \neq 0$ such that*

$$a_0 + a_1 f + a_2 f^2 + \cdots + a_m f^m = 0. \tag{3.1}$$

Then f is a polynomial of degree no more than $\max(\deg a_0, \ldots, \deg a_m)$.

Remark. Take $\deg 0 = -\infty$.

Proof. Let f, m, and $a_0, \ldots, a_m$ be as in the hypothesis of the lemma, and let $N = \max(\deg a_0, \ldots, \deg a_m)$. We claim that $f(z)/z^{N+1} \to 0$ as $|z| \to \infty$. By way of contradiction, suppose that there exists $\epsilon > 0$ and a sequence $\{z_i\}_{i=0}^{\infty}$ such that $|z_i| \to \infty$ as $i \to \infty$ and $|f(z_i)/z_i^{N+1}| > \epsilon$ for all i. It follows that $|f(z_i)| \to \infty$ as $i \to \infty$. Solving (3.1) for f, we see that

$$f = (-a_0/f^{m-1} - a_1/f^{m-2} - \cdots - a_{m-1})/a_m.$$

Dividing both sides by z^{N+1}, we have

$$f/z^{N+1} = (-a_0/(z^{N+1}f^{m-1}) - a_1/(z^{N+1}f^{m-2}) - \cdots - a_{m-1}/z^{N+1})/a_m. \tag{3.2}$$

Notice that each $a_j(z_i)/z_i^{N+1} \to 0$ as $i \to \infty$. Since $|a_m(z_i)|$ and $|f(z_i)|$ tend to infinity as $i \to \infty$, the right-hand side of (3.2) evaluated at z_i tends to zero as $i \to \infty$. This is a contradiction, and thus f/z^{N+1} tends to zero as z tends to infinity. It follows from Cauchy's integral formula that f is a polynomial of degree no more than N. □

The following generalization will be used in the sequel.

Lemma 3.2. *Suppose $f: \mathbb{C}^j \times \mathbb{C}^k \to \mathbb{C}$ is holomorphic and algebraic over the set of functions*

$$B = \{g: \mathbb{C}^j \times \mathbb{C}^k \to \mathbb{C} : g \text{ is holomorphic and polynomial in its last } k \text{ variables}\}.$$

Then f is in B.

Proof. Let f and B be as described in the hypothesis of the lemma and choose $a_0, \ldots, a_m$ in B such that $m > 0$, $a_m \neq 0$, and

$$a_0 + a_1 f + a_2 f^2 + \cdots + a_m f^m = 0.$$

Let $\deg a_i$ denote the degree of a_i in its last k variables. Let $(\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k)$ be in the nonempty open set $a_m^{-1}(\mathbb{C} - 0)$. It follows from Lemma 3.1 that for each i between 1 and k the function

$$h(z) = f(\alpha_1, \ldots, \alpha_j, \beta_1, \ldots, z, \ldots, \beta_k)$$

obtained by replacing β_i with z is polynomial of degree no more than the degrees

$$N = \max(\deg a_0, \ldots, \deg a_m).$$

We write

$$f = f(t_1, \dots, t_j, x_1, \dots, x_k).$$

Then for i between 1 and k we have $(\partial/\partial x_i)^{N+1} f = 0$ on $a_m^{-1}(\mathbb{C} - 0)$. Hence $(\partial/\partial x_i)^{N+1} f = 0$ at every point of $\mathbb{C}^j \times \mathbb{C}^k$. Thus f is polynomial in its last k variables. □

4. Algebraic closure of $T(D)$ in $\mathbb{C}^{[n]}$

Throughout this section, let ϕ be the flow associated with (3.1), let D be the derivation associated with $\mathbf{V}$, and let $T(D)$ be the corresponding torsion part of $\mathbb{C}^{[n]}$. Our main theorem follows.

Theorem 4.1. *If ϕ is holomorphic on $\mathbb{C} \times \mathbb{C}^n$, then $T(D)$ is algebraically closed in $\mathbb{C}^{[n]}$.*

Proof. Let $b \in \mathbb{C}^{[n]}$ be algebraic over $T(D)$. Then there exist an integer $m > 0$ and polynomials $a_1, \dots, a_m$ in $T(D)$ with $a_m \neq 0$ such that

$$a_m b^m + a_{m-1} b^{m-1} + \cdots + a_0 = 0.$$

Thus

$$a_m|_\phi (b|_\phi)^m + a_{m-1}|_\phi (b|_\phi)^{m-1} + \cdots + a_0|_\phi = 0.$$

That is, by Lemma 2.1 $b|_\phi$ is holomorphic on $\mathbb{C} \times \mathbb{C}^n$ and algebraic over the set of functions holomorphic on $\mathbb{C} \times \mathbb{C}^n$ and polynomial in their last n variables. By Lemma 3.1, $b|_\phi$ is polynomial in its last n variables. By Lemma 2.1, b is in $T(D)$. Hence $T(D)$ is algebraically closed in $\mathbb{C}^{[n]}$. □

Example. Consider the initial value problem

$$\dot{x} = x \qquad x(0) = u$$
$$\dot{y} = xy \qquad y(0) = v$$

which has flow $\phi(t, (u, v)^T) = (ue^t, ve^{u(e^t - 1)})^T$. Notice that ϕ is holomorphic on $\mathbb{C} \times \mathbb{C}^n$. Furthermore, the first component of ϕ is polynomial in the initial condition while the second component is not. That is, X_1 is in $T(D)$ while X_2 is not. Since $T(D)$ is algebraically closed in $\mathbb{C}^{[2]}$ and

$$\mathbb{C}[X_1] \subset T(D) \subsetneqq \mathbb{C}[X_1, X_2],$$

we must have $T(D) = \mathbb{C}[X_1]$.

5. A connection to the Jacobian conjecture

A polynomial map $F: \mathbb{C}^n \to \mathbb{C}^n$ is said to be a *polynomial automorphism* if $F = F(X_1, \dots, X_n) = (F_1, \dots, F_n)$ has a polynomial inverse. The Jacobian conjecture, attributed to Keller [9], asserts that if the Jacobian matrix $JF = (\partial F_i/\partial X_j)$ of F has a nonzero constant determinant, then the map has a polynomial inverse. At present the conjecture is open for $n > 1$. See Bass, Connell, and Wright [2] for a discussion of the Jacobian conjecture.

As the following theorem shows, determining when the torsion part of $\mathbb{C}^{[n]}$ with respect to the action of a derivation is algebraically closed in $\mathbb{C}^{[n]}$ may shed light on the Jacobian conjecture.

Theorem 5.1. *Suppose* $F: \mathbb{C}^n \to \mathbb{C}^n$ *is a polynomial mapping satisfying*

$$\det JF = 1 \quad \text{and} \quad F(0) = 0. \tag{5.1}$$

Let D *be the derivation associated with the vector field*

$$\mathbf{V}(y) = -[JF(y)]^{-1}F(y). \tag{5.2}$$

Then F *is a polynomial automorphism if and only if* $T(D)$ *is algebraically closed in* $\mathbb{C}^{[n]}$.

Remarks. Zampieri discusses (5.2) in the case where F is a continuously differentiable function defined on some subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ in connection with the invertibility of F. Notice that if $H: \mathbb{C}^n \to \mathbb{C}^n$ and $\det JH$ is a nonzero constant, then $F = [JH(0)]^{-1}(H - H(0))$ satisfies (5.1).

Proof. Let ϕ be the flow associated with

$$\dot{y} = -[JF(y)]^{-1}F(y). \tag{5.3}$$

Notice that $DF = -F$ and thus, as noted by Zampieri [16],

$$F(\phi(t, x)) = F(x)e^{-t}. \tag{5.4}$$

Thus each component of F is in $T(D)$ and since $T(D)$ is a ring containing $\mathbb{C}$, it follows that $\mathbb{C}[F_1, \dots, F_n]$ is contained in $T(D)$.

If F is a polynomial automorphism, then $\mathbb{C}[F_1, \dots, F_n] = \mathbb{C}^{[n]}$. Hence that $T(D)$ is algebraically closed in $\mathbb{C}^{[n]}$ follows trivially since $T(D) = \mathbb{C}^{[n]}$. To prove the converse, notice that since $F_1, \dots, F_n$ are algebraically independent, each X_i must be algebraic over $\mathbb{C}(F_1, \dots, F_n)$ and hence algebraic over $\mathbb{C}[F_1, \dots, F_n]$. Thus each X_i must be algebraic over $T(D)$. By hypothesis, $T(D)$ is algebraically closed in $\mathbb{C}^{[n]}$. It follows that $T(D) = \mathbb{C}^{[n]}$. Thus by Theorem 3.1 of Coomes and Zurkowski [8], equation (5.3) has a polynomial flow

$$\phi(t, x) = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i} a_{ij}(x) t^j e^{\lambda_i t}, \qquad (t, x) \in \mathbb{C} \times \mathbb{C}^n, \tag{5.5}$$

where each a_{ij} is a vector whose components are polynomial. Notice that

$$J(-[JF]^{-1}F)(0) = -I.$$

From this and Corollary 4.1 of [8] we may assume that each λ_i is a nonpositive integer.

Since $\det JF = 1$ and $F(0) = 0$, there exist neighborhoods U and V of 0 and an analytic function $G: V \to U$ such that for all x in U and y in V, we have

$$G(F(x)) = x \quad \text{and} \quad F(G(y)) = y. \tag{5.6}$$

It follows from (5.4) and (5.6) that for each x in U, the flow $\phi(t, x)$ tends to zero as t tends to infinity along the real axis. Thus we may assume each λ_i is negative. If we can show that for each $R > 0$ the mapping F is invertible on

$$B(0, R) = \{z \in \mathbb{C}^n : |z| < R\}$$

it will follow that F is injective and hence has a polynomial inverse by Theorem 2.1 of Bass, Connell, and Wright [2].

We proceed in a fashion similar to Zampieri [16]. It follows from (5.4) and (5.5) that there exists T_R such that for all x in $B(0, R)$

$$F(\phi(T_R, x)) \in V \quad \text{and} \quad \phi(T_R, x) \in U.$$

To see that F is invertible on $B(0, R)$, notice that for all x in $B(0, R)$

$$\phi^{T_R}(x) = G(F(\phi(T_R, x)) = G(F(x)e^{-T_R})$$

and hence

$$x = \phi^{-T_R}(G(F(x)e^{-T_R})).$$

That is, if $M_{e^{-T_R}}$ is the map given by multiplication by e^{-T_R}, then $\phi^{-T_R} \circ G \circ M_{e^{-T_R}}$ is left inverse for $F|_{B(0,R)}$. In particular, F is injective on $B(0, R)$. Thus F is a polynomial automorphism. □

Since the polynomials $F_1, \ldots, F_n$ are algebraically independent and contained in $T(D)$, it follows that $T(D)$ is algebraically closed in $\mathbb{C}^{[n]}$ if and only if $T(D) = \mathbb{C}^{[n]}$ and $T(D) = \mathbb{C}^{[n]}$ if and only if ϕ is a polynomial flow.

Acknowledgment

The author wishes to express his thanks to E. H. Connell for the discussions that lead to this paper.

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