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TWISTED TORUS BUNDLES OVER ARITHMETIC VARIETIES

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ABSTRACT. A twisted torus is a nilmanifold which is the quotient of a real Heisenberg group by a cocompact discrete subgroup. We construct fiber bundles over arithmetic varieties whose fibers are isomorphic to a twisted torus, and express the complex cohomology of such bundles over certain Riemann surfaces in terms of automorphic forms.

1. INTRODUCTION

Heisenberg groups are certain two-step nilpotent groups and they play an important role in quantum mechanics, harmonic analysis, and many other areas of mathematics (see e.g. [2], [3], [5], [10], [13]). A twisted torus is a nilmanifold which is the quotient of a real Heisenberg group by a cocompact discrete subgroup. In this paper we consider fiber bundles over arithmetic varieties whose fibers are isomorphic to a twisted torus.

Let V be a vector space of dimension $2m$ over $\mathbb{Q}$ and let β be a nondegenerate alternating bilinear form on V. If $V(\mathbb{R})$ denotes the real vector space $V \otimes_{\mathbb{Q}} \mathbb{R}$, then the real Heisenberg group $H(\mathbb{R}, \beta)$ associated to β is the product $V(\mathbb{R}) \times \mathbb{R}$ with multiplication given by

$$(v_1, c_1) \cdot (v_2, c_2) = (v_1 + v_2, c_1 + c_2 + \beta(v_1, v_2)/2)$$

for $(v_1, c_1), (v_2, c_2) \in V(\mathbb{R}) \times \mathbb{R}$. Let $Sp(V(\mathbb{R}), \beta)$ be the symplectic group determined by the bilinear form β on $V(\mathbb{R})$, and let $\mathscr{H}_m$ be the Siegel upper half-space on which $Sp(V(\mathbb{R}), \beta)$ operates. We define a group structure on the product $Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ with multiplication given by

$$(g_1, v_1, c_1) \cdot (g_2, v_2, c_2) = (g_1 g_2, v_1 + g_1 v_2, c_1 + c_2 + \beta(v_1, g_1 v_2)/2)$$

for all $(g_1, v_1, c_1), (g_2, v_2, c_2) \in Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$. Then the group $Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ acts on $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$ by

$$(g, v, c) \cdot (z, w, d) = (gz, v + gw, c + d + \beta(v, gw)/2)$$

for $(g, v, c) \in Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ and $(z, w, d) \in \mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$.

Let G be a semisimple algebraic group defined over $\mathbb{Q}$, and let K be a maximal compact subgroup of the semisimple Lie group $G(\mathbb{R})$. We assume

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that the symmetric space $D = G(\mathbb{R})/K$ has a $G(\mathbb{R})$-invariant complex structure. Let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free cocompact arithmetic subgroup of G, and let $X = \Gamma\backslash D$ be the corresponding arithmetic variety. Let L be a lattice in $V(\mathbb{R})$ such that $\beta(L, L) \subset \mathbb{Z}$, and let Γ_0 be a torsion-free subgroup of $Sp(L, \beta)$ of finite index. Let $\rho : G(\mathbb{R}) \to Sp(V(\mathbb{R}), \beta)$ be a symplectic representation of $G(\mathbb{R})$, and let $\tau : D \to \mathscr{H}_m$ be a holomorphic map such that $\rho(\Gamma) \subset \Gamma_0$ and

$$\tau(gy) = \rho(g)\tau(y) \quad \text{for all} \quad g \in G(\mathbb{R}) \quad \text{and} \quad y \in D.$$

Then the group $\Gamma_0 \times L \times \mathbb{Z}$ operates on the space $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$ properly discontinuously. We denote by W_0 the quotient space $\Gamma_0 \times L \times \mathbb{Z}\backslash\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$. Then the natural projection $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R} \to \mathscr{H}_m$ induces a fiber bundle $\pi_0 : W_0 \to X_0$ over the complex manifold $X_0 = \Gamma_0\backslash\mathscr{H}_m$. If $\tau_X : X \to X_0$ is the map induced by τ, then we obtain a twisted torus bundle $\pi_W : W \to X$ by pulling back the bundle $\pi_0 : W_0 \to X_0$ by the map $\tau_X : X \to X_0$, and the fibers of π_W are isomorphic to the twisted torus $H(\mathbb{Z}, \beta)_L\backslash H(\mathbb{R}, \beta)$, where $H(\mathbb{Z}, \beta)_L = L \times \mathbb{Z}$.

In this paper, we express the complex cohomology of W in terms of certain automorphic forms with respect to Γ when $X = \Gamma\backslash\mathscr{H}$ is a compact Riemann surface associated to a quaternion algebra.

2. Twisted torus bundles

In this section, we construct a twisted torus bundle $\pi : W \to X$ over an arithmetic variety $X = \Gamma\backslash D$ whose fibers are isomorphic to the quotient $H(\mathbb{Z}, \beta)_L\backslash H(\mathbb{R}, \beta)$ of the real Heisenberg group $H(\mathbb{R}, \beta)$ by a cocompact discrete subgroup $H(\mathbb{Z}, \beta)_L$.

First, we shall construct a universal bundle $\pi_0 : W_0 \to X_0$. Let V be a $\mathbb{Q}$-vector space of dimension $2n > 0$, and let β be a nondegenerate alternating bilinear form on V. Let $Sp(V, \beta)$ be the symplectic group determined by β, i.e.,

$$Sp(V, \beta) = \{g \in GL(V) \mid \beta(gu, gv) = \beta(u, v) \text{ for all } u, v \in V\}.$$

Let $V(\mathbb{R})$ be the real vector space $V \otimes_{\mathbb{Q}} \mathbb{R}$, and let $Sp(V(\mathbb{R}), \beta)$ be the corresponding symplectic group. We define a group structure on the product $Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ with multiplication given by

$$(g_1, v_1, c_1) \cdot (g_2, v_2, c_2) = (g_1 g_2, v_1 + g_1 v_2, c_1 + c_2 + \beta(v_1, g_1 v_2)/2)$$

for all $(g_1, v_1, c_1), (g_2, v_2, c_2) \in Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$. Let $\mathscr{H}_m$ be the Siegel upper half-space

$$\mathscr{H}_m = \{J \in GL(V(\mathbb{R})) \mid J^2 = -1, \quad \beta(x, Jy) \text{ is a positive definite symmetric bilinear form in } x, y \in V(\mathbb{R})\}$$

on which $Sp(V(\mathbb{R}), \beta)$ operates.

Lemma 2.1. *The group $Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ operates on $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$ by*

$$(g, v, c) \cdot (z, w, d) = (gz, v + gw, c + d + \beta(v, gw)/2)$$

for $(g, v, c) \in Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ and $(z, w, d) \in \mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$.

Proof. Let $(g_1, v_1, c_1), (g_2, v_2, c_2) \in Sp(V(\mathbb{R}), \beta) \times V(\mathbb{R}) \times \mathbb{R}$ and $(z, w, d) \in \mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$. Then we have

$$\begin{aligned}
&\Big((g_1, v_1, c_1) \cdot (g_2, v_2, c_2)\Big) \cdot (z, w, d) \\
&\quad = \Big(g_1 g_2, v_1 + g_1 v_2, c_1 + c_2 + \beta(v_1, g_1 v_2)/2\Big) \cdot (z, w, d) \\
&\quad = \Big(g_1 g_2 z, v_1 + g_1 v_2 + g_1 g_2 w, \\
&\qquad c_1 + c_2 + \beta(v_1, g_1 v_2)/2 + d + \beta(v_1 + g_1 v_2, g_1 g_2 w)/2\Big) \\
&\quad = \Big(g_1 g_2 z, v_1 + g_1 v_2 + g_1 g_2 w, \\
&\qquad c_1 + c_2 + d + \big(\beta(v_1, g_1 v_2) + \beta(v_1, g_1 g_2 w) + \beta(v_2, g_2 w)\big)/2\Big);
\end{aligned}$$

here we used the relation $\beta(g_1 v_2, g_1 g_2 w) = \beta(v_2, g_2 w)$. On the other hand we have

$$\begin{aligned}
&(g_1, v_1, c_1) \cdot \Big((g_2, v_2, c_2) \cdot (z, w, d)\Big) \\
&\quad = (g_1, v_1, c_1)\Big(g_2 z, v_2 + g_2 w, c_2 + d + \beta(v_2, g_2 w)/2\Big) \\
&\quad = \Big(g_1 g_2 z, v_1 + g_1(v_2 + g_2 w), \\
&\qquad c_1 + c_2 + d + \beta(v_2, g_2 w)/2 + \beta(v_1, g_1(v_2 + g_2 w))/2\Big) \\
&\quad = \Big(g_1 g_2 z, v_1 + g_1 v_2 + g_1 g_2 w, \\
&\qquad c_1 + c_2 + d + \big(\beta(v_2, g_2 w) + \beta(v_1, g_1 v_2) + \beta(v_1, g_1 g_2 w)\big)/2\Big).
\end{aligned}$$

Hence the lemma follows. □

Let L be a lattice in $V(\mathbb{R})$ such that $\beta(L, L) \subset \mathbb{Z}$, and let Γ_0 be a torsion-free subgroup of $Sp(L, \beta)$ of finite index, where

$$Sp(L, \beta) = \{g \in Sp(V(\mathbb{R}), \beta) \mid gL = L\}.$$

Then the group $\Gamma_0 \times L \times \mathbb{Z}$ operates on the space $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$ properly discontinuously. We denote by W_0 the quotient space $\Gamma_0 \times L \times \mathbb{Z} \backslash \mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R}$. Then the natural projection $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R} \to \mathscr{H}_m$ induces a fiber bundle $\pi_{W,0} : W_0 \to X_0$ over the complex manifold $X_0 = \Gamma_0 \backslash \mathscr{H}_m$.

Let $H(\mathbb{R}, \beta)$ denote the real Heisenberg group associated to β. Thus $H(\mathbb{R}, \beta) = V(\mathbb{R}) \times \mathbb{R}$ with its multiplication operation given by

$$(v, c) \cdot (w, d) = (v + w, c + d + \beta(v, w)/2).$$

Let $H(\mathbb{Z}, \beta)_L$ be the subgroup $L \times \mathbb{Z} \subset V(\mathbb{R}) \times \mathbb{R}$ of $H(\mathbb{R}, \beta)$. Then each fiber of π_0 is isomorphic to the quotient $H(\mathbb{Z}, \beta)_L \backslash H(\mathbb{R}, \beta)$, which is a circle bundle over a torus called a *twisted torus*.

Let G be a semisimple algebraic group defined over $\mathbb{Q}$, and let K be a maximal compact subgroup of the semisimple Lie group $G(\mathbb{R})$. We assume that the symmetric space $D = G(\mathbb{R})/K$ has a $G(\mathbb{R})$-invariant complex structure. Let $\Gamma \subset G(\mathbb{Q})$ be a torsion-free cocompact arithmetic subgroup G. Then the quotient $X = \Gamma \backslash D$ has a structure of a complex projective variety called an arithmetic variety (see e.g. [6]). Let $\rho : G \to Sp(V, \beta)$ be a symplectic representation and $\tau : D \to \mathscr{H}_m$ a holomorphic map such that $\rho(\Gamma) \subset \Gamma_0$ and

$$\tau(gy) = \rho(g)\tau(y) \quad \text{for all} \quad g \in G(\mathbb{R}) \quad \text{and} \quad y \in D.$$

If $\tau_X : X \to X_0$ is the map induced by τ, then we obtain a map $\pi_W : W \to X$ by pulling back the bundle W_0 via the map $\tau_X : X \to X_0$. Thus we have the following commutative diagram:

$$\begin{tikzcd} \tau_X^*(W_0) = W \arrow[r] \arrow[d, "\pi_W"'] & W_0 \arrow[d, "\pi_{W,0}"] \\ X \arrow[r, "\tau_X"] & X_0 \end{tikzcd}$$

The map $\pi_W : W \to X$ is a fiber bundle, called a *twisted torus bundle*, and each of its fibers is isomorphic to the twisted torus $H(\mathbb{Z}, \beta)_L \backslash H(\mathbb{R}, \beta)$. This bundle can also be described as follows: Let $W = \Gamma \times L \times \mathbb{Z} \backslash D \times V(\mathbb{R}) \times \mathbb{R}$, where $\Gamma \times L \times \mathbb{Z}$ acts on $D \times V(\mathbb{R}) \times \mathbb{R}$ by

$$(\gamma, l, c) \cdot (z, w, d) = (\rho(\gamma)z, l + \rho(\gamma)w, c + d + \beta(l, \rho(g)w)/2)$$

for all $(\gamma, l, c) \in \Gamma \times L \times \mathbb{Z}$ and $(z, w, d) \in D \times V(\mathbb{R}) \times \mathbb{R}$. Then the twisted torus bundle $\pi_W : W \to X$ is the fiber bundle induced by the natural projection

$$\Gamma \times L \times \mathbb{Z} \backslash D \times V(\mathbb{R}) \times \mathbb{R} \to \Gamma \backslash D = X.$$

3. Kuga fiber varieties

In this section, we review the construction of Kuga fiber varieties over arithmetic varieties whose fibers are polarized abelian varieties. Let $V(\mathbb{R})$, β, L, $\mathscr{H}_m$ and Γ_0 as in §2. Then each element $J \in \mathscr{H}_m$ defines a complex structure on $V(\mathbb{R})$ and there is a unique complex analytic structure on $\mathscr{H}_m \times V(\mathbb{R})$ such that the projection $P : \mathscr{H}_m \times V(\mathbb{R}) \to \mathscr{H}_m$ becomes a complex vector bundle over $\mathscr{H}_m$. For each J, if we denote the complex vector space $(V(\mathbb{R}), J)$ by $V(\mathbb{R})_J$, then the complex torus $Y_J = V(\mathbb{R})_J/L$ is an abelian variety that has a polarization determined by β. We set

$$Y_{\mathscr{H}_m} = L \backslash \mathscr{H}_m \times V(\mathbb{R}),$$

where the action of L on $\mathscr{H}_m \times V(\mathbb{R})$ is given by

$$l \cdot (J, v) = (J, v + l) \quad \text{for} \quad J \in \mathscr{H}_m, v \in V(\mathbb{R}) \quad \text{and} \quad l \in L.$$

Then the vector bundle $P : \mathscr{H}_m \times V(\mathbb{R}) \to \mathscr{H}_m$ induces the fiber bundle $\pi_{\mathscr{H}_m} : Y_{\mathscr{H}_m} \to \mathscr{H}_m$ whose fibers are abelian varieties polarized by β. Then the quotient $X_0 = \Gamma_0 \backslash \mathscr{H}_m$ is an arithmetic variety that can be considered as a Zariski open subset of a complex projective variety. Now the fiber bundle $\pi_{\mathscr{H}_m} : Y_{\mathscr{H}_m} \to \mathscr{H}_m$ induces a universal family of abelian varieties $\pi_{Y,0} : Y_0 \to X_0$ over X_0 described in Example 1 in [11, §IV.7].

Let G, K, $D = G(\mathbb{R})/K$, $\Gamma \subset G(\mathbb{Q})$, $X = \Gamma \backslash D$, $\rho : G(\mathbb{R}) \to Sp(V(\mathbb{R}), \beta)$ and $\tau : D \to \mathscr{H}_m$ be as in §2. If $\tau_X : X \to X_0$ is the map induced by $\tau : D \to \mathscr{H}_m$, then by pulling back the bundle $\pi_{Y,0} : Y_0 \to X_0$ via the map τ_X we obtain the *Kuga fiber variety* $\pi_Y : Y \to X$ over the arithmetic variety $X = \Gamma \backslash D$. Kuga fiber varieties can also be described as follows: The semidirect product $\Gamma \ltimes_\rho L$

with respect to the representation $\rho : \Gamma \to \operatorname{Aut}(L)$ operates on the product manifold $D \times V(\mathbb{R})$ properly discontinuously by

$$(\gamma, l) \cdot (y, v) = (\gamma y, \rho(\gamma)v + l)$$

for $(\gamma, l) \in \Gamma \ltimes_\rho L$ and $(y, v) \in D \times V(\mathbb{R})$, and this action coincides with the restriction of the operation of $\Gamma \times L \times \mathbb{Z}$ on $D \times V(\mathbb{R}) \times \mathbb{R}$ described at the end of §2 to the group $\Gamma \times L \subset \Gamma \times L \times \mathbb{Z}$. We set

$$Y = \Gamma \ltimes_\rho L \backslash D \times V(\mathbb{R}),$$

and denote by π the natural projection of Y onto $X = \Gamma \backslash D$. It is known that Y has a structure of a complex projective variety and that the fiber Y_x over each $x \in X$ is an abelian variety isomorphic to the quotient $V(\mathbb{R})/L$ polarized by β (see [7], [8], [9], [11, Chapter 4] for details).

4. The cohomology

Let $\pi_W : W \to X$ be the fiber bundle constructed in §2 and let $\pi_Y : Y \to X$ be the Kuga fiber variety as in §2. Since the restriction of the operation of $\Gamma \times L \times \mathbb{Z}$ on $D \times V(\mathbb{R}) \times \mathbb{R}$ described in §2 to $\Gamma \times L$ coincides with the action of $\Gamma \ltimes_\rho L$ on $D \times V(\mathbb{R})$ in §3, the natural projection $\mathscr{H}_m \times V(\mathbb{R}) \times \mathbb{R} \to \mathscr{H}_m \times V(\mathbb{R})$ induces a fiber bundle $\pi_{W,Y} : W \to Y$ whose fibers are isomorphic to the circle $\mathbb{R}/\mathbb{Z}$. Thus W can be considered as a circle bundle over the Kuga fiber variety Y. In this section, we express the complex cohomology $H^*(W, \mathbb{C})$ of the twisted torus bundle W in terms of the cohomology $H^*(Y, \mathbb{C})$ of the Kuga fiber variety Y.

Theorem 4.1 (Hochschild-Serre). *For any group extension*

$$1 \to \Phi \to \mathfrak{G} \to \Delta \to 1$$

and any $\mathfrak{G}$-module $\mathscr{M}$, there is a spectral sequence $\{\mathscr{E}_r\}$ with $\mathscr{E}_2$-term

$$\mathscr{E}_2^{pq} = H^p(\Delta, H^q(\Phi, \mathscr{M}))$$

that converges to the cohomology $H^*(\mathfrak{G}, \mathscr{M})$.

Proof. See [4] or [1, §VII.6]. □

Now we apply the Hochschild-Serre theorem to the groups $\mathfrak{G} = \Gamma \times L \times \mathbb{Z}$, $\Delta = \Gamma \times L$, the abelian group $\Phi = \mathbb{Z}$, and the trivial $\Gamma \times L \times \mathbb{Z}$-module $\mathscr{M} = \mathbb{C}$. Thus we have a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = H^p(\Gamma \times L, H^q(\mathbb{Z}, \mathbb{C}))$$

and

$$H^r(\Gamma \times L \times \mathbb{Z}, \mathbb{C}) \cong \bigoplus_{p+q=r} E_\infty^{p,q}.$$

Lemma 4.2. *The cohomology $H^q(\mathbb{Z}, \mathbb{C})$ of the abelian group $\mathbb{Z}$ with coefficients in the trivial $\mathbb{Z}$-module $\mathbb{C}$ is given as follows:*

$$H^0(\mathbb{Z}, \mathbb{C}) = H^1(\mathbb{Z}, \mathbb{C}) = \mathbb{C}, \quad H^q(\mathbb{Z}, \mathbb{C}) = 0 \textit{ for } q \neq 0, 1.$$

Proof. The homology $H_1(\mathbb{Z}, \mathbb{Z})$ of the group $\mathbb{Z}$ with coefficients in the trivial $\mathbb{Z}$-module $\mathbb{Z}$ is isomorphic to $\mathbb{Z}/[\mathbb{Z}, \mathbb{Z}] = \mathbb{Z}$. Hence we have

$$H_1(\mathbb{Z}, \mathbb{C}) = \mathbb{Z} \otimes \mathbb{C} = \mathbb{C}, \quad H^1(\mathbb{Z}, \mathbb{C}) = \mathbb{C}^*$$

and

$$H^q(\mathbb{Z}, \mathbb{C}) = \bigwedge^q(H^1(\mathbb{Z}, \mathbb{C})) = \bigwedge^q(\mathbb{C}^*),$$

where $\mathbb{C}^*$ denotes the dual of the trivial $\mathbb{Z}$-module $\mathbb{C}$. Since $\bigwedge^q(\mathbb{C}^*) = \mathbb{C}$ for $q = 0, 1$ and $\bigwedge^q(\mathbb{C}^*) = 0$ for $q \neq 0, 1$, the proof is complete. □

From the relation $H^q(\mathbb{Z}, \mathbb{C}) = \bigwedge^q(\mathbb{C}^*)$ given in the proof of Lemma 4.2, it follows that the spectral sequence $\{E_r\}$ has the E_2-term

$$E_2^{p,q} = H^p(\Gamma \times L, H^p(\mathbb{Z}, \mathbb{C})) = H^q(\Gamma \times L, \bigwedge^q(\mathbb{C}^*)).$$

Proposition 4.3. *The spectral sequence $\{E_r\}$ degenerates at E_2.*

Proof. For each positive integer a, the map $\mu_a : \Gamma \times L \times \mathbb{Z} \to \Gamma \times L \times \mathbb{Z}$ defined by

$$\mu_a(\gamma, l, k) = (\gamma, al, a^2k)$$

for $(\gamma, l, k) \in \Gamma \times L \times \mathbb{Z}$ is a homomorphism of groups. Hence μ_a induces maps

$$\mu_a^* : H^r(\Gamma \times L \times \mathbb{Z}, \mathbb{C}) \to H^r(\Gamma \times L \times \mathbb{Z}, \mathbb{C})$$

and $(\mu_a^*)_r^{p,q} : E_r^{p,q} \to E_r^{p,q}$ such that

$$d_r \circ (\mu_a^*)_r^{p,q} = (\mu_a^*)_r^{p+r,q-r+1} \circ d_r,$$

where $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is the boundary map. Since $E_2^{p,q} = H^q(\Gamma \times L, \bigwedge^q(\mathbb{C}^*))$, the map $(\mu_a^*)_2^{p,q}$ can be considered as the multiplication by a^{2q}. Hence we have

$$d_2(a^{2q}v) = a^{2q-2}(d_2(v))$$

for all $v \in E_2^{p,q}$. Since a is an arbitrary positive integer, it follows that $d_2 = 0$. Similarly, we obtain $d_s = 0$ for all $s \geq 2$. Hence $\{E_r\}$ degenerates at E_2 and the proof is complete. □

Theorem 4.4. *The cohomology $H^r(Y, \mathbb{C})$ with coefficient $\mathbb{C}$ has the decomposition*

$$H^r(W, \mathbb{C}) = H^r(Y, \mathbb{C}) \oplus H^{r-1}(Y, \mathbb{C})$$

for $1 \leq r \leq \dim_{\mathbb{R}} W$.

Proof. Since we have

$$W = \Gamma \times L \times \mathbb{Z} \backslash D \times V(\mathbb{R}) \times \mathbb{R} \quad \text{and} \quad Y = \Gamma \times L \backslash D \times V(\mathbb{R}),$$

the group cohomologies $H^r(\Gamma \times L \times \mathbb{Z}, \mathbb{C})$ and $H^r(\Gamma \times L, \mathbb{C})$ with coefficients in the trivial module $\mathbb{C}$ can be identified with the complex cohomologies $H^r(W, \mathbb{C})$ and $H^r(Y, \mathbb{C})$ respectively. Using Proposition 4.3 and Lemma 4.2, we have

$$\begin{aligned} H^r(W, \mathbb{C}) = H^r(\Gamma \times L \times \mathbb{Z}, \mathbb{C}) &= \bigoplus_{p+q=r} E_2^{p,q} = \bigoplus_{p+q=r} H^p(\Gamma \times L, \bigwedge^q(\mathbb{C}^*)) \\ &= H^r(\Gamma \times L, \mathbb{C}) \oplus H^{r-1}(\Gamma \times L, \mathbb{C}) = H^r(Y, \mathbb{C}) \oplus H^{r-1}(Y, \mathbb{C}). \end{aligned}$$

Hence the theorem follows. □

5. AUTOMORPHIC FORMS

In this section we consider twisted torus bundles and Kuga fiber varieties of a special type. Let G be the semisimple algebraic $\mathbb{Q}$-group SL_2 so that $G(\mathbb{R}) = SL(2, \mathbb{R})$, $K = SO(2)$, and D is the Poincaré upper half-plane $\mathscr{H}$. Let B be an indefinite quaternion algebra over $\mathbb{Q}$ such that $B \otimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R})$, and let $\mathscr{O}$ be a maximal order in B. Given a positive integer N, let Γ_N be the discrete subgroup of $SL(2, \mathbb{R})$ defined by

$$\Gamma_N = \{\alpha \in \mathscr{O} \mid \alpha\mathscr{O} = \mathscr{O}, \quad \det\alpha = 1, \quad \alpha = 1 \pmod{N\mathscr{O}}\}.$$

Then the quotient space $\Gamma_N \backslash \mathscr{H}$ is a compact Riemann surface. We set

$$V_m = M(2, \mathbb{R})^m, \quad L_m = \mathscr{O}^m,$$

and define the representation $\rho_m : G(\mathbb{R}) \to GL(V_m)$ by

$$\rho_m(g)(\mu_1, \dots, \mu_m) = (g\mu_1, \dots, g\mu_m)$$

for all $(\mu_1, \dots, \mu_m) \in M(2, \mathbb{R})^m$. Let the twisted torus bundle $\pi_W : W \to X$ and the Kuga fiber variety $\pi_Y : Y \to X$ be the ones constructed from $D = \mathscr{H}$, $\Gamma = \Gamma_N$, $L = L_m$ and $\rho = \rho_m$.

Since $X = \Gamma\backslash\mathscr{H} = \Gamma_N\backslash\mathscr{H}$ is compact, the group Γ has no cusps in $\mathscr{H}$; hence any holomorphic function $f : \mathscr{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = (cz+d)^k F(z) \quad \text{for} \quad z \in \mathscr{H}, \quad \gamma \in \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \subset SL(2, \mathbb{R})$$

is both an automorphic form and a cusp form of weight k with respect to Γ (see e.g. [12]). We denote by $S_k(\Gamma)$ the space of all automorphic forms of weight k with respect to Γ.

Theorem 5.1. *Let $\pi_W : W \to X$ be the twisted torus bundle over $X = \Gamma\backslash\mathscr{H}$ associated to an indefinite quaternion algebra B as described above, and let ω_0 be the cohomology class of the two-form $y^{-2}(dz \wedge d\bar{z})$ in $H^2(W, \mathbb{C})$. Then, for each $r \geq 3$, the complex cohomology $H^r(W, \mathbb{C})$ of W has the following decomposition:*

$$\begin{aligned} H^r(W, \mathbb{C}) = {} & A_r\left(S_{r+1}(\Gamma) \oplus \overline{S_{r+1}(\Gamma)}\right) \\ & \oplus \left(\bigoplus_{j=0}^{r-2} B_{r,j}\left(S_{j+2}(\Gamma) \oplus \overline{S_{j+2}(\Gamma)}\right)\right) \oplus C_r\mathbb{C} \oplus C_{r-2}\mathbb{C}\omega_0, \end{aligned}$$

where $A_r = \binom{2m}{r-1}$,

$$B_{r,j} = \begin{cases} \binom{2m}{r-j_0-1}\binom{2m}{j_0-1} - \binom{2m}{r-j_0}\binom{2m}{j_0-2} & \text{if } j = r-2j_0, \\[2ex] \binom{2m}{r-j_0-1}\binom{2m}{j_0} - \binom{2m}{r-j_0}\binom{2m}{j_0-1} & \text{if } j = r-2j_0-1, \end{cases}$$

and

$$C_r = \binom{2m}{r_0}^2 - \binom{2m}{r_0+1}\binom{2m}{r_0-1} \quad \text{if } r = 2r_0 \text{ or } 2r_0+1.$$

Proof. By Equation (11) in [7, §IV.2] or Equation (1) in [8] we have a decomposition of the cohomology $H^p(Y, \mathbb{C})$ of the Kuga fiber variety Y of the form

$$H^p(Y, \mathbb{C}) = a(p, 0)\mathbb{C} \oplus \left(\bigoplus_{j=0}^{p-1} a(p-1, j)\left(S_{j+2}(\Gamma) \oplus \overline{S_{j+2}(\Gamma)}\right)\right) \oplus a(p-2, 0)\mathbb{C}\omega_0,$$

where

$$a(s, t) = \binom{2m}{(s+t)/2}\binom{2m}{(s-t)/2} - \binom{2m}{(s+t)/2+1}\binom{2m}{(s-t)/2-1}$$

if $s \equiv t \pmod 2$, and $a(s, t) = 0$ if $s \not\equiv t \pmod 2$. Using Theorem 4.4, we have

$$\begin{aligned} H^r(W, \mathbb{C}) = {} & \left(a(r, 0) + a(r-1, 0)\right)\mathbb{C} \\ & \oplus \left(\bigoplus_{j=0}^{r-2}\left(a(r-1, j) + a(r-2, j)\right)\left(S_{j+2}(\Gamma) \oplus \overline{S_{j+2}(\Gamma)}\right)\right) \\ & \oplus a(r-1, r-1)\left(S_{r+1}(\Gamma) \oplus \overline{S_{r+1}(\Gamma)}\right) \\ & \oplus \left(a(r-2, 0) + a(r-3, 0)\right)\mathbb{C}\omega_0. \end{aligned}$$

The formula for A_r follows from $a(r-1, r-1) = \binom{2m}{r-1}$. The formula for $B_{r,j}$ is obtained from the relations

$$a(r-1, j) + a(r-2, j) = \begin{cases} a(r-2, j) = a(r-2, r-2j_0) & \text{if } j = 2 - 2j_0, \\ a(r-1, j) = a(r-1, r-2j_0-1) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } j = r - 2j_0 - 1. \end{cases}$$

For C_r, we have $a(r, 0) + a(r-1, 0) = a(r, 0) = a(2r_0, 0)$ if $r = 2r_0$, and $a(r, 0) + a(r-1, 0) = a(r-1, 0) = a(2r_0, 0)$ if $r = 2r_0 + 1$. Hence we obtain the formula for C_r, and the proof is complete. □

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