PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 8, August 1995

# THE MINIMUM NORM OF CERTAIN COMPLETELY POSITIVE MAPS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $L$ be a completely bounded linear map from a unital $C^*$-algebra to the algebra of all bounded linear operators on a Hilbert space. Then

$$\min\left\{\|\phi\|_{\mathrm{cb}}:\begin{pmatrix}\phi & L & 0 & \cdots & 0\\ & & & & \vdots\\ L^* & \phi & L & & 0\\ 0 & & \ddots & & \\ \vdots & & & & L\\ 0 & \cdots & 0 & L^* & \phi\end{pmatrix}_{n\times n}\ \begin{matrix}\text{is completely}\\ \text{positive}\\ \text{for all } n\end{matrix}\right\}=2S(L),$$

where $S(L)=\min\{\|\phi\|_{\mathrm{cb}}\colon \phi\pm\operatorname{Re}\lambda L$ is completely positive for all $|\lambda|=1\}$.

## 1. INTRODUCTION

Let $M_n$ denote the $C^*$-algebra of complex $n\times n$ matrices generated by the matrix units $E_{ij}$ $(i, j = 1, 2, 3, \ldots, n)$ and $B(H)$ the algebra of all bounded linear operators on a Hilbert space $H$. Let $A$ and $B$ be $C^*$-algebras, and let $L\colon A\to B$ be a bounded linear map; the map $L$ is called completely positive if $L\otimes I_n\colon A\otimes M_n\to B\otimes M_n$ defined by $L\otimes I_n(a\otimes b)=L(a)\otimes b$ is positive for all $n$. $L$ is completely bounded if $\sup_n\|L\otimes I_n\|$ is finite, and we write $\|L\|_{\mathrm{cb}}=\sup_n\|L\otimes I_n\|$. We define $L^*(a)=L(a^*)^*$ and $S'$ the commutant of $S$ contained in $B(H)$. Let $T$ be a bounded linear operators on $H$; the numerical radius of $T$ is $w(T)=\sup_{\|h\|=1}\{|\langle Th, h\rangle|\}$.

In [3, Theorem 2.2], it has been shown that every completely bounded map $L$ from a unital $C^*$-algebra $A$ into $B(H)$ has a minimal commutant representation $L(a)=V^*T\Pi(a)V$. In [4, Theorem 2.7], we have

$$\begin{aligned} S(L) &= \min\{\|\phi\|_{\mathrm{cb}}\colon \phi\pm\operatorname{Re}\lambda L \text{ is completely positive for all } |\lambda|=1\}\\ &= \min\{w(T)\colon L \text{ has a minimal commutant representation } V^*T\Pi V\}.\end{aligned}$$

The work in this paper is to sharpen the estimate in the above mentioned theorems. Example 2.8 shows that there exists some case such that

$$S(L)<\|L\|_{\mathrm{cb}}<\min\left\{\|\phi\|_{\mathrm{cb}}\colon\begin{pmatrix}\phi & L\\ L^* & \phi\end{pmatrix}\text{ is completely positive}\right\}<2S(L).$$

Received by the editors March 5, 1993 and, in revised form, April 12, 1993.
1991 *Mathematics Subject Classification*. Primary 46L05.
*Key words and phrases.* Completely positive map, completely bounded map, numerical radius.

Hence, if

$$\begin{pmatrix} \phi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \phi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \phi \end{pmatrix}_{n\times n}$$

is completely positive for all $n$, then $\left(\begin{smallmatrix} \phi & L \\ L^* & \phi \end{smallmatrix}\right)$ is completely positive. However, the converse is not true. If $\left(\begin{smallmatrix} \phi & L \\ L^* & \phi \end{smallmatrix}\right)$ is completely positive, then $\phi \pm \lambda L$ is completely positive for all $|\lambda| = 1$. The converse is also not true.

## 2. The numerical radius

Applying the results from Paulsen [2], we have the following properties.

**Proposition 2.1.** *The following three statements are equivalent*:

(1) $w(T) \leq 1$.

(2) $$\begin{pmatrix} 2 & T & T^2 & \dots & T^n \\ & & & & \vdots \\ T^* & 2 & T & & \\ T^2 & & \ddots & & T \\ T^{n^*} & \dots & & T^* & 2 \end{pmatrix} \geq 0 \quad \textit{for all } n.$$

(3) $$\begin{pmatrix} 2 & T & 0 & \dots & 0 \\ T^* & 2 & T & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & T \\ 0 & \dots & 0 & T^* & 2 \end{pmatrix}_{n\times n} \geq 0 \quad \textit{for all } n.$$

*Proof*. Using the results from Paulsen [2, p. 36], (1) implies (2).

Suppose that (2) is true. Let

$$R = \begin{pmatrix} 0 & T & 0 & \dots & 0 \\ & & & & \vdots \\ 0 & 0 & T & & 0 \\ 0 & 0 & 0 & & \\ \vdots & & & \ddots & T \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

then

$$\begin{pmatrix} 2 & T & T^2 & \dots & T^n \\ & & & & \\ T^* & 2 & T & & \\ T^{*^2} & T^* & \ddots & & \\ \vdots & & & & T \\ T^{*^n} & \dots & & T^* & 2 \end{pmatrix} = 1 + R + R^2 + \dots + R^n + 1 + R^* + \dots + R^{*^n}.$$

Since

$$\begin{aligned}
&\langle (1+R+R^2+\cdots+R^n+1+R^*+\cdots+R^{*^n})(1-R)y\,,\,(1-R)y\rangle \\
&\quad = \langle (2-R-R^*)y\,,\,y\rangle \\
&\quad = \left\langle \begin{pmatrix} 2 & -T & 0 & \dots & 0 \\ & & & & \vdots \\ -T^* & 2 & & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & -T \\ 0 & \dots & 0 & -T^* & 2 \end{pmatrix} y\,,\, y \right\rangle \geq 0
\end{aligned}$$

for all $y$ in $H^{n+1}$ and for all $n$, we have (3).

If (3) is true, we claim that $w(T) \leq 1/\cos(\frac{\pi}{n+1})$ for all $n$.

Let

$$P_n^c = \begin{pmatrix} 2 & cT & 0 & \dots & 0 \\ & & & & \vdots \\ \bar{c}T^* & 2 & cT & & 0 \\ 0 & & 2 & & \\ \vdots & & & \ddots & cT \\ 0 & \dots & 0 & \bar{c}T^* & 2 \end{pmatrix}_{n\times n};$$

then

$$P_n^c = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & & & & \vdots \\ 0 & \bar{c} & 0 & & 0 \\ 0 & 0 & \bar{c}^2 & & 0 \\ \vdots & & & \ddots & \\ 0 & \dots 0 & 0 & & \bar{c}^n \end{pmatrix} \begin{pmatrix} 2 & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & 2 & T & & 0 \\ 0 & T^* & 2 & & \\ \vdots & & & \ddots & T \\ 0 & \dots & 0 & T^* & 2 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & & & & \vdots \\ 0 & c & 0 & & 0 \\ 0 & 0 & c^2 & & 0 \\ \vdots & & & \ddots & \\ 0 & \dots 0 & 0 & & c^n \end{pmatrix} \geq 0$$

for $|c| = 1$ and for all $n$. Let $m$ be a fixed number,

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{m+1} \end{pmatrix} \quad \text{with } \|\lambda\| = 1,$$

$$S_{m+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & & & & \vdots \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \\ \vdots & & & \ddots & 1 \\ 0 & \dots 0 & 0 & & 0 \end{pmatrix}_{(m+1)\times(m+1)},$$

and $h_i = \lambda_i h$ for $i = 1, 2, 3, \dots, m+1$, where $h$ is in $H$ with $\|h\| = 1$, then

$$(*) \qquad \left\langle P_{m+1}^c \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m+1} \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{m+1} \end{pmatrix} \right\rangle = 2 + 2\,\mathrm{Re}(c\langle S_{m+1}\lambda, \lambda\rangle \langle Th, h\rangle) \geq 0.$$

Similarly,

$$(**) \qquad 2 - 2\,\mathrm{Re}(c\langle S_{m+1}\lambda, \lambda\rangle \langle Th, h\rangle) \geq 0.$$

From $(*)$ and $(**)$, we have $|\mathrm{Re}\, c\langle S_{m+1}\lambda, \lambda\rangle\langle Th, h\rangle| \leq 1$ for all $|c| = 1$. Hence, $|\langle S_{m+1}\lambda, \lambda\rangle\langle Th, h\rangle| \leq 1$ or

$$w(T) \leq 1/w(S_{m+1}) = 1/\cos\tfrac{\pi}{m+2} \quad [1, \text{Proposition } 1].$$

Therefore, (3) implies (1).

**Proposition 2.2.**

$$\min\left\{\rho : \begin{pmatrix} \rho & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & \rho & T & & 0 \\ 0 & & & & \\ \vdots & & & \ddots & T \\ 0 & \dots & 0 & T^* & \rho \end{pmatrix}_{n\times n} \geq 0 \quad \textit{for nonnegative number } \rho \textit{ and for all } n \right\}$$

$= 2w(T)$.

*Proof*. Suppose that

$$\begin{pmatrix} \rho & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & \rho & T & & 0 \\ 0 & & & & \\ \vdots & & & \ddots & T \\ 0 & \dots & 0 & T^* & \rho \end{pmatrix}_{n\times n} \geq 0$$

for some $\rho \geq 0$ and for all $n$. Applying Proposition 2.1, we have $w(2T/\rho) \leq 1$ or $w(2T) \leq \rho$. Since $w(T/w(T)) = 1$, we have

$$\begin{pmatrix} 2w(T) & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & 2w(T) & T & & 0 \\ 0 & & & & \\ \vdots & & & \ddots & T \\ 0 & \dots & 0 & T^* & 2w(T) \end{pmatrix}_{n\times n} \geq 0 \quad \text{for all } n.$$

**Corollary 2.3.** *If*

$$\begin{pmatrix} \rho & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & \rho & T & & 0 \\ 0 & & & & \\ \vdots & & & \ddots & T \\ 0 & \dots & 0 & T^* & \rho \end{pmatrix}_{n\times n} \geq 0$$

*and* $w(T) > \rho/2$, *then* $n \leq [\pi/\cos^{-1}(\frac{\rho}{2w(T)}) - 1]$, *where* $[x]$ *is the greatest integer less than or equal to* $x$. *Moreover,* $\max\{m: w(T) \leq \rho/2w(S_m)\} = [\pi/\cos^{-1}(\frac{\rho}{2w(T)}) - 1]$.

*Proof.* Since

$$\begin{pmatrix} 2 & 2T/\rho & 0 & \dots & 0 \\ & & & & \vdots \\ 2T^*/\rho & 2 & 2T/\rho & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & 2T/\rho \\ 0 & \dots & 0 & 2T^*/\rho & 2 \end{pmatrix}_{n\times n}$$

is positive, from the proof of Proposition 2.1, we have

$$w\left(\frac{2T}{\rho}\right) \leq \frac{1}{w(S_n)} = \frac{1}{\cos(\pi/(n+1))}.$$

Let $m_0 = \max\{m: w(2T/\rho) \leq 1/w(S_m)\}$; then $w(2T/\rho) \leq 1/w(S_{m_0})$. Hence $n \leq m_0 \leq \pi/\cos^{-1}(\frac{\rho}{2w(T)}) - 1$. Now, $w(2T/\rho) > 1/w(S_{m_0+1})$, we have $\pi/\cos^{-1}(\frac{\rho}{2w(T)}) - 2 < m_0$. Therefore, $m_0 = [\pi/\cos^{-1}(\frac{\rho}{2w(T)}) - 1]$, where $[x]$ is the greatest integer less than or equal to $x$.

**Example 2.4.** From [1, Proposition 1] and Proposition 2.2, we have

$$\min\left\{\rho: \begin{pmatrix} \rho & S_{m+1} & 0 & \dots & 0 \\ & & & & \vdots \\ S_{m+1}^* & \rho & S_{m+1} & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & S_{m+1} \\ 0 & \dots & 0 & S_{m+1}^* & \rho \end{pmatrix}_{n\times n} \geq 0 \text{ for all } n\right\}$$

$$= 2w(S_{m+1}) = 2\cos\frac{\pi}{m+2}$$

and $\|S_m\| = 1$. Let

$$x = \frac{1}{\sqrt{m+1}}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

then

$$w(S_{m+1}) = \cos\tfrac{\pi}{m+2} \geq \langle S_{m+1}x, x\rangle = \tfrac{m}{m+1}.$$

If $m > 1$, then $2w(S_{m+1}) \geq \frac{2m}{m+1} > 1$. Hence,

$$\begin{pmatrix} 1 & S_{m+1} & 0 & \dots & 0 \\ & & & & \vdots \\ S_{m+1}^* & 1 & S_{m+1} & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & S_{m+1} \\ 0 & \dots & 0 & S_{m+1}^* & 1 \end{pmatrix}_{n\times n} \text{ is not positive}$$

when $n > \pi/\cos^{-1}(1/2w(S_{m+1})) - 1$.

**Corollary 2.5.** *If*

$$\begin{pmatrix} 2 & T & T^2 & \dots & T^n \\ & & & & \vdots \\ T^* & 2 & T & & T^{n-1} \\ T^{2*} & & \ddots & & \\ \vdots & & & & T \\ T^{n*} & \dots & T^{n-1*} & T^* & 2 \end{pmatrix}$$

*is positive and $w(T) > 1$, then $n \leq 1[\pi/\cos^{-1}(\frac{1}{2w(T)}) - 1]$, where $[x]$ is the greatest integer less than or equal to $x$.*

*Proof.* From the proof of Proposition 2.1, we have

$$\begin{pmatrix} 2 & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & 2 & T & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & T \\ 0 & \dots & 0 & T^* & 2 \end{pmatrix}_{n\times n} \geq 0.$$

Applying Corollary 2.3, we have the corollary.

**Theorem 2.6** [4, Theorem 2.7]. *Let $L: A \to B(H)$ be a completely bounded linear map. Then*

$$\begin{aligned} S(L) &= \min\{\|\phi\|_{cb} : \phi \pm \operatorname{Re} cL \text{ is completely positive for all } |c| = 1\} \\ &= \min\{w(T) : L \text{ has a minimal commutant representation } V^*T\Pi V\}. \end{aligned}$$

**Theorem 2.7.**

$$\min\left\{ \|\phi\|_{cb} : \begin{pmatrix} \phi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \phi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \phi \end{pmatrix}_{n\times n} \begin{array}{l} \text{is completely} \\ \text{positive for all } n \end{array} \right\} = 2S(L).$$

*Proof*. If

$$\begin{pmatrix} \phi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \phi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \phi \end{pmatrix}_{n\times n}$$

is completely positive for some $\phi$ and for all $n$, then there exists a completely positive map $\psi$ with $\psi(I) = \|\phi\|_{\text{cb}} I$ such that

$$\begin{pmatrix} \psi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \psi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \psi \end{pmatrix}_{n\times n}$$

is completely positive. Since $\left(\begin{smallmatrix} \psi & L \\ L^* & \psi \end{smallmatrix}\right)$ is completely positive, applying [3, Theorem 2.2], we have $\psi = \|\phi\|_{\text{cb}} V^* \sqcap V$ and $L = \|\phi\|_{\text{cb}} V^* T \sqcap V$, where $V$ is an isometry, $\sqcap$ is a $*$-representation, and $T$ is in $\sqcap(A)'$. Applying [3, Proposition 2.6], we have

$$\|\phi\|_{\text{cb}} \begin{pmatrix} 1 & T & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & 1 & T & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & T \\ 0 & \dots & 0 & T^* & 1 \end{pmatrix}_{n\times n} \geq 0.$$

Applying Proposition 2.2, we have $2w(T) \leq 1$. Using Theorem 2.6, we have

$$2S(L) \leq 2w(\|\phi\|_{\text{cb}} T) \leq \|\phi\|_{\text{cb}}.$$

Conversely, applying [4, Theorem 2.7], there exist an isometry $\overline{V}$, a $*$-representation $\overline{\prod}$, and a bounded linear operator $\overline{T}$ in $\overline{\prod}(A)'$ such that $L = \overline{V}^* \overline{T}\,\overline{\prod}\,\overline{V}$ with $w(\overline{T}) = S(L)$. Since

$$\begin{pmatrix} 2w(\overline{T}) & \overline{T} & 0 & \dots & 0 \\ & & & & \vdots \\ \overline{T}^* & 2w(\overline{T}) & \overline{T} & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & \overline{T} \\ 0 & \dots & 0 & \overline{T}^* & 2w(\overline{T}) \end{pmatrix}_{n\times n} \geq 0 \quad \text{for all } n,$$

applying [3, Proposition 2.6], we have that

$$\begin{pmatrix} 2w(\overline{T})\overline{V}^*\overline{\prod V} & \overline{V}^*\overline{T\prod V} & 0 & \dots & 0 \\ & & & & \vdots \\ \overline{V}^*\overline{T\prod V} & 2w(\overline{T})\overline{V}^*\overline{\prod V} & \overline{V}^*\overline{T\prod V} & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & \dots & 0 & \overline{V}^*\overline{T}^*\overline{\prod V} & 2w(\overline{T})\overline{V}^*\overline{\prod V} \end{pmatrix}_{n\times n}$$

is completely positive. Hence we have proved the theorem.

**Example 2.8.** Let $L\colon C\oplus C\to M_2(C)$ be defined by

$$a\oplus b\to\frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ b & 0\end{pmatrix}.$$

Let the completely positive map $\psi\colon C\oplus C\to M_2(C)$ be defined by

$$a\oplus b\to\frac{1}{\sqrt{2}}\begin{pmatrix} a+b & 0 \\ 0 & a+b\end{pmatrix}.$$

It is not difficult to see that $\left(\begin{smallmatrix}\psi & L\\ L^* & \psi\end{smallmatrix}\right)$ is completely positive and

$$\|\psi\|_{\text{cb}}=\sqrt{2}=\min\left\{\|\phi\|_{\text{cb}}\colon\begin{pmatrix}\phi & L\\ L^* & \phi\end{pmatrix}\text{ is completely positive}\right\}.$$

Since $\|L\|_{\text{cb}}=1$ and $S(L)=(1+\sqrt{2})/2\sqrt{2}$ [4, Example 2.8], we have

$$S(L)<\|L\|_{\text{cb}}<\|\psi\|_{\text{cb}}<2S(L)$$

$$=\min\left\{\|\phi\|_{\text{cb}}\colon\begin{pmatrix}\phi & L & 0 & \dots & 0\\ & & & & \vdots\\ L^* & \phi & L & & 0\\ 0 & & \ddots & & \\ \vdots & & & & L\\ 0 & \dots & 0 & L^* & \phi\end{pmatrix}_{n\times n}\begin{array}{l}\text{is completely}\\ \text{positive for all } n\end{array}\right\}.$$

**Corollary 2.9.** *If $L$ is a self-adjoint completely bounded linear map, then*

$$2\|L\|_{\text{cb}}=\min\left\{\|\phi\|_{\text{cb}}\colon\begin{pmatrix}\phi & L & 0 & \dots & 0\\ & & & & \vdots\\ L & \phi & L & & 0\\ 0 & & \ddots & & \\ \vdots & & & & L\\ 0 & \dots & 0 & L & \phi\end{pmatrix}_{n\times n}\begin{array}{l}\textit{is completely}\\ \textit{positive for all } n\end{array}\right\}.$$

*Proof.* Applying [4, Corollary 2.9], we have $S(L)=\|L\|_{\text{cb}}$.

**Corollary 2.10.** *If $L$ is a bounded linear functional, then*

$$2\|L\| = \min\left\{ \|\phi\|_{cb} : \begin{pmatrix} \phi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \phi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \phi \end{pmatrix}_{n\times n} \text{ is completely positive for all } n \right\}.$$

*Proof.* Since $\|L\| = S(L)$ [4, Corollary 3.3], we have the corollary.

**Corollary 2.11.** *If*

$$\begin{pmatrix} \phi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \phi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \phi \end{pmatrix}_{n\times n}$$

*is completely positive with $n > 1$ and $0 < \|\phi\|_{cb} < 2S(L)$, then*

$$n \le \left[\frac{\pi}{\cos^{-1}(\|\phi\|_{cb}/2S(L))} - 1\right],$$

*where $[x]$ is the greatest integer less than or equal to $x$.*

*Proof.* From the proof of the Theorem 2.7, we have

$$w(\|\phi\|_{cb}T) \ge S(L) > \frac{1}{2}\|\phi\|_{cb} \quad \text{and} \quad \begin{pmatrix} 1 & T^* & 0 & \dots & 0 \\ & & & & \vdots \\ T^* & 1 & T & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & T \\ 0 & \dots & 0 & T^* & 1 \end{pmatrix}_{n\times n} \ge 0.$$

Since $w(T) > 1/2$ and $1/2w(T) \le \|\phi\|_{cb}/2S(L) < 1$, applying Corollary 2.3, we obtain

$$n \le \left[\frac{\pi}{\cos^{-1}(1/2w(T))} - 1\right] \le \left[\frac{\pi}{\cos^{-1}(\|\phi\|_{cb}/2S(L))} - 1\right].$$

**Example 2.12.** If $\psi$ and $L$ are from Example 2.8 and

$$\begin{pmatrix} \psi & L & 0 & \dots & 0 \\ & & & & \vdots \\ L^* & \psi & L & & 0 \\ 0 & & \ddots & & \\ \vdots & & & & L \\ 0 & \dots & 0 & L^* & \psi \end{pmatrix}_{n\times n}$$

is completely positive, then

$$n \leq \left[ \frac{\pi}{\cos^{-1}(2/1+\sqrt{2}))} - 1 \right] = 4 .$$

## ACKNOWLEDGMENT

The author gratefully acknowledges several valuable conversations with Professor Vern Paulsen.

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