

## A CONSTRAINT ON THE EXISTENCE OF SIMPLE TORSION-FREE LIE MODULES

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(Communicated by Maurice Auslander)

**ABSTRACT.** For any simple Lie algebra  $L$  with Cartan subalgebra  $H$  the classification of all simple  $H$ -diagonalizable  $L$ -modules having a finite-dimensional weight space is known to depend on determining the simple torsion-free  $L$ -modules of finite degree. It is further known that the only simple Lie algebras which admit simple torsion-free modules of finite degree are those of types  $A_n$  and  $C_n$ . For the case of  $A_n$  we show that there are no simple torsion-free  $A_n$ -modules of degree  $k$  for  $n \geq 4$  and  $2 \leq k \leq n - 2$ . We conclude with some examples showing that there exist simple torsion-free  $A_n$ -modules of degrees  $1$ ,  $n - 1$ , and  $n$ .

Let  $L$  be a finite-dimensional simple Lie algebra over the complex numbers  $\mathbb{C}$ , and  $H$  be a fixed Cartan subalgebra of  $L$ . An  $H$ -diagonalizable  $L$ -module  $M$  is said to be: of *degree*  $k$  provided each weight space of  $M$  is of dimension  $k$ ; *torsion-free* provided each root vector of  $L$  acts on each weight space of  $M$  in a one-to-one manner; and *pointed* provided it is simple and has a 1-dimensional weight space.

Theorem 4.18 of [F] describes simple  $H$ -diagonalizable  $L$ -modules having a finite-dimensional weight space as the amalgam of highest weight modules and torsion-free modules, thereby reducing the classification of such modules to determining the simple torsion-free modules of degree  $k < \infty$ . The complete classification of pointed modules was presented in [BL1]. In studying modules of minimal weight space dimension greater than 1, we must study simple torsion-free modules of degree greater than 1. In [F], Fernando showed that only  $A_n$ ,  $n \geq 1$ , and  $C_n$ ,  $n \geq 2$ , possess torsion-free modules. It is easy to see that no simple torsion-free modules of minimal weight space dimension greater than 1 exist for  $A_1$ . The existence of simple torsion-free  $A_n$ -modules of arbitrary degree for  $n = 2, 3$  has been established in [BBL].

In this note, we further restrict the existence of torsion-free  $A_n$ -modules by proving

**Main Theorem.** *There are no simple torsion-free  $A_n$ -modules of degree  $k$  for  $n \geq 4$  and  $2 \leq k \leq n - 2$ .*

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Received by the editors October 30, 1991 and, in revised form, November 11, 1993.

1991 *Mathematics Subject Classification.* Primary 17B10.

The first author was supported in part by NSERC Grant #A8471.

The second author was supported in part by NSERC Grant #A7742.

We consider  $A_n$  to be the  $(n + 1) \times (n + 1)$  traceless matrices over  $\mathbb{C}$ . Let  $\{E_{ij} | 1 \leq i, j \leq n + 1\}$  be a standard set of matrix units and  $\{\varepsilon_i | 1 \leq i \leq n + 1\}$  be a standard basis of  $\mathbb{C}^{n+1}$ . Naturally, we think of  $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} | 1 \leq i \leq n\}$  as a base of the root system  $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) | 1 \leq i < j \leq n\}$  of  $A_n$ . If  $i \leq j$  and  $\alpha = \alpha_i + \dots + \alpha_j = \varepsilon_i - \varepsilon_{j+1}$ , then the root vectors  $X_\alpha, X_{-\alpha}$  are  $E_{i, j+1}, E_{j+1, i}$ , respectively, and  $h_\alpha = E_{ii} - E_{j+1, j+1}$ .  $H = \text{span}_{\mathbb{C}}\{h_\alpha | \alpha \in \Delta\}$  is our fixed Cartan subalgebra. In subsequent sections we will have occasion to use other simple bases for the root system  $\Phi$  of  $A_n$ . In all cases we order the simple basis elements to preserve the connectedness of adjacent roots.

Let  $U = U(A_n)$  be the universal enveloping algebra of  $A_n$  and  $U_0 = U_0(A_n) = \{u \in U | [u, h] = 0, \forall h \in H\}$ . Using Theorem 2.1 of [BL2], the elements  $u \in U_0$  can be characterized as linear combinations of products of  $h_\alpha, \alpha \in \Delta$ , and elements of the form

$$(1) \quad u = E_{i_1, i_2} \cdots E_{i_{l-1}, i_l} E_{i_l, i_1},$$

where  $i_1, \dots, i_l$  are distinct in  $\{1, \dots, n + 1\}$ .

In the torsion-free setting that we are working in it is easy to see that  $M$  is a simple  $U$ -module if and only if for any weight  $\Theta$   $M_\Theta$  is a simple  $U_0$ -module with  $M = UM_\Theta$ . This means that we can restrict our attention to an arbitrary weight space  $M_\Theta$  and so, at our convenience, we can choose  $\Theta$  such that it satisfies the General Assumption below.

The Main Theorem is proved by a sequence of lemmas under the following

**General Assumption.**  $M$  is a torsion-free  $A_n$ -module of degree  $k < \infty$  and  $n \geq 2$ . Using the torsion-free assumption we fix a weight  $\Theta$  of  $M$  such that  $\Theta(h_\alpha) \neq 0$  for all  $\alpha \in \Phi$ . Denote the restriction of  $X_{-\alpha}X_\alpha$  to the weight space  $M_\Theta$  by  $\mathcal{A}_\alpha$ .

Let  $i, j, m$  be any three distinct values in  $\{1, \dots, n + 1\}$ . Set

$$\begin{aligned} c_1(i, j, m) &= E_{ji}E_{ij}, & c_2(i, j, m) &= E_{mj}E_{jm}, & c_3(i, j, m) &= E_{mi}E_{im}, \\ c_4(i, j, m) &= E_{mi}E_{ij}E_{jm}, & c_5(i, j, m) &= E_{mj}E_{ji}E_{im}, \\ h_1(i, j, m) &= E_{ii} - E_{jj}, & \text{and } h_2(i, j, m) &= E_{jj} - E_{mm}. \end{aligned}$$

Drop  $(i, j, m)$  to simplify notation. One can show that the following identities hold in  $U$ :

$$\begin{aligned} (2) \quad & [c_1, c_2] = [c_2, c_3] = -[c_1, c_3] = c_5 - c_4, \\ (3) \quad & [c_1, c_4] = -c_2c_1 + c_3c_1 - c_4h_1 + c_3h_1 - c_5 + c_4, \\ (4) \quad & [c_2, c_4] = c_2c_1 - c_3c_2 - c_4h_2 + c_5 - c_4, \\ (5) \quad & [c_1, [c_1, c_2]] = 2c_2c_1 - 2c_3c_1 + (c_5 + c_4)h_1 - 2c_3h_1 + 2[c_1, c_2]. \end{aligned}$$

Let  $\tilde{c}_i$  denote the action of  $c_i$  on  $M_\Theta$  and  $\tilde{h}_i = \Theta(h_i)$ . We use  $W \leq M_\Theta$  to indicate that  $W$  is a subspace of  $M_\Theta$ .

**Lemma 6.** *If  $W \leq M_\Theta$  is invariant under  $\mathcal{A}_\alpha$  for all  $\alpha \in \Phi$ , then  $W$  is a  $U_0$ -submodule of  $M_\Theta$ .*

*Proof.* It suffices to show that  $W$  is invariant under all  $u$  of form (1). By hypothesis, this is true if  $l = 2$ . Assume  $l = 3$  so that  $u = c_4(i, j, m) = E_{mi}E_{ij}E_{jm}$ . Since  $\Theta(h_1) \neq 0$ , we can use (2) and (5) to solve for  $\tilde{c}_4$  and  $\tilde{c}_5$

and see that  $\tilde{c}_4$  and  $\tilde{c}_5$  leave  $W$  invariant. Assume that  $W$  is invariant under all elements of form (1) with  $l < m$ , and observe that in  $U_0$

$$(7) \quad \begin{aligned} & (E_{i_1, i_2} \cdots E_{i_{m-1}, i_m} E_{i_m, i_1})(E_{i_{m-1}, i_1} E_{i_1, i_{m-1}}) \\ & = (E_{i_1, i_2} \cdots E_{i_{m-2}, i_{m-1}} E_{i_{m-1}, i_1})(E_{i_{m-1}, i_m} E_{i_m, i_1} E_{i_1, i_{m-1}}). \end{aligned}$$

Since the two expressions of form (1) on the right-hand side of (10) leave  $W$  invariant as does the invertible operator  $E_{i_{m-1}, i_1} E_{i_1, i_{m-1}}$ , the result follows by induction.  $\square$

*Notation.* For each  $1 \leq m \leq n$ , let  $\Phi^m$  be the root subsystem generated by  $\Delta^m = \{\alpha_m, \dots, \alpha_n\} \subseteq \Delta$ . If  $\alpha, \beta, \alpha + \beta \in \Phi$ , then  $A(\alpha, \beta)$  denotes the  $A_2$  subalgebra of  $A_n$  generated by  $X_{\pm\alpha}$  and  $X_{\pm\beta}$ .

If  $\mathcal{A}_\alpha$  acts like a complex multiple of the identity map on  $W \leq M_\Theta$ , then we say that  $\mathcal{A}_\alpha$  is a scalar on  $W$ .

**Lemma 8.** *Let  $\alpha, \beta, \alpha + \beta \in \Phi$  and  $W \leq M_\Theta$ . Then*

(i) *if  $\mathcal{A}_\alpha$  is a scalar on  $W$  and  $W$  is invariant under  $\mathcal{A}_\beta$ , then  $W$  is a  $U_0(A(\alpha, \beta))$ -module,*

(ii) *if  $\mathcal{A}_\alpha$  and  $\mathcal{A}_\beta$  are both scalars on  $W$ , then all elements in  $U_0(A(\alpha, \beta))$  are scalars on  $W$ , and*

(iii) *if  $\mathcal{A}_\alpha$  is a scalar on  $W$  and  $\mathcal{A}_\beta$  is a scalar on  $W$  for all  $\beta \in \Phi$  such that  $\alpha + \beta \in \Phi$ , then  $\mathcal{A}_\gamma$  is a scalar on  $W$  for all  $\gamma \in \Phi$ .*

*Proof.* Let  $w \in W$  and  $\mathcal{A}_\alpha$  be a scalar on  $W$ . Without loss of generality, we may assume that  $X_\alpha = E_{ij}$  and  $X_\beta = E_{jm}$ . Then with  $\tilde{c}_1 = \mathcal{A}_\alpha$  and  $\tilde{c}_2 = \mathcal{A}_\beta$  we have that  $0 = [\tilde{c}_1, \tilde{c}_2]w = (\tilde{c}_5 - \tilde{c}_4)w$  and so, by (2),  $[\tilde{c}_r, \tilde{c}_s]w = 0$  for  $1 \leq r, s \leq 3$ . By (2) and (5),  $\tilde{c}_4$  and  $\tilde{c}_5$  are expressible in terms of  $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ . Therefore,

$$(9) \quad 0 = [\tilde{c}_1, \tilde{c}_4]w = (-\tilde{c}_2\tilde{c}_1 + \tilde{c}_3\tilde{c}_1 - \tilde{c}_4\tilde{h}_1 + \tilde{c}_3\tilde{h}_1)w,$$

$$(10) \quad 0 = [\tilde{c}_2, \tilde{c}_4]w = (\tilde{c}_2\tilde{c}_1 - \tilde{c}_3\tilde{c}_2 - \tilde{c}_4\tilde{h}_2)w,$$

and we have

$$(11) \quad \tilde{c}_2\tilde{c}_1(\tilde{h}_1 + \tilde{h}_2)w = \tilde{c}_3(\tilde{c}_1\tilde{h}_2 + \tilde{c}_2\tilde{h}_1 + \tilde{h}_1\tilde{h}_2)w.$$

Since, if  $w \neq 0$ , the left-hand side of equality (11) is a nonzero element in  $W$ ,  $(\tilde{c}_1\tilde{h}_2 + \tilde{c}_2\tilde{h}_1 + \tilde{h}_1\tilde{h}_2)$  is an invertible operator on  $W$  and we see that  $\tilde{c}_3$  leaves  $W$  invariant, (i) follows from this and Lemma 6. Also, if  $\mathcal{A}_\beta$  is a scalar on  $W$ , then (11) implies  $\tilde{c}_3$  is a scalar on  $W$  and (10) implies that  $\tilde{c}_4$  is and (ii) follows.

Now for (iii), without loss of generality, take  $\alpha_1 = \alpha$  in  $\Delta$ . Clearly,  $\mathcal{A}_{\alpha_2}$  and  $\mathcal{A}_{-(\alpha_1+\alpha_2+\alpha_3)}$  are scalars on  $W$ . Part (ii) implies  $\mathcal{A}_{\alpha_1+\alpha_2}$  is a scalar on  $W$ .  $\mathcal{A}_{\alpha_3}$  is a scalar on  $W$  since it denotes the action of an element of

$$U_0(A(\alpha_1 + \alpha_2, -(\alpha_1 + \alpha_2 + \alpha_3))).$$

Continuing in this fashion one gets that  $\mathcal{A}_{\alpha_i}$  is a scalar on  $W$  for all  $\alpha_i \in \Delta$ , and the result follows from part (ii).  $\square$

**Lemma 12.** Fix an eigenvalue  $\lambda_i$  of  $\mathcal{A}_{\alpha_i}$  for each  $\alpha_i \in \Delta$ , and let  $W_i$  be the corresponding eigenspace. For each  $1 \leq j \leq n$ , define  $W^{(j-1)} = W_1 \cap \dots \cap W_{j-1}$ . If  $\mathcal{A}_{\alpha_j}$  is a scalar on  $W^{(j-1)} \neq 0$  and  $k > 1$ , then  $M$  is not simple. (In particular, if for some  $\beta \in \Phi$   $\mathcal{A}_\beta$  is a scalar on  $M_\Theta$ , then  $M$  is not simple.)

*Proof.* Assume that  $M$  is simple of degree  $k \geq 2$ .  $W^{(j-1)} \neq 0$  is invariant under  $\mathcal{A}_\alpha$  for each  $\alpha \in \Phi^{j+1}$  because  $\mathcal{A}_\alpha \mathcal{A}_{\alpha_i} = \mathcal{A}_{\alpha_i} \mathcal{A}_\alpha$  for  $1 \leq i \leq j-1$ . Lemma 8(i) implies  $W^{(j-1)}$  is invariant under  $\mathcal{A}_\alpha$  for  $\alpha \in \Phi^j$  and in turn for  $\alpha \in \Phi^{j-1}$ ,  $\alpha \in \Phi^{j-2}$ ,  $\dots$ ,  $\alpha \in \Phi^1 = \Phi$ . If  $W^{(j-1)} \neq M_\Theta$ , then by Lemma 6,  $W^{(j-1)}$  is a proper  $U_0$ -submodule of  $M_\Theta$ , which implies that  $M$  is not simple.

If  $W^{(j-1)} = M_\Theta$ , then let  $l$  be the smallest subscript such that  $\mathcal{A}_{\alpha_l}$  is not a scalar on  $M_\Theta$ . By Lemma 6 and Lemma 8(ii), such an  $l$  must exist. Note that  $\Delta' = \{\alpha'_1 = \alpha_l, \alpha'_2 = \alpha_{l-1}, \dots, \alpha'_l = \alpha_1, \alpha'_{l+1} = -(\alpha_{l+1} + \alpha_l + \dots + \alpha_1), \alpha'_{l+2} = -\alpha_{l+2}, \dots, \alpha'_n = -\alpha_n\}$  is a base such that if we let  $W'_1$  be an eigenspace corresponding to  $\mathcal{A}_{\alpha'_1}$ , then  $\mathcal{A}_{\alpha'_2}$  is a scalar on  $M_\Theta \neq W'^{(1)} = W'_1 \neq 0$ . We are now back to the case handled above.  $\square$

**Lemma 13.** Let  $n \geq 4$ . If the degree  $k$  of  $M$  satisfies  $2 \leq k \leq n-2$  and  $v$  is an eigenvector of  $\mathcal{A}_\alpha$  for each  $\alpha \in \Delta^3$ , then  $M$  is not simple.

*Proof.* Suppose that  $M$  is simple, and for  $3 \leq i \leq n$  let  $\lambda_i$  be the eigenvalue of  $\mathcal{A}_{\alpha_i}$  belonging to  $v$ . Let  $W_i$  be the eigenspace of  $\mathcal{A}_{\alpha_i}$  belonging to  $\lambda_i$ . By Lemma 12,  $\dim W_i < k \leq n-2$ . Since we can complete  $\{\alpha'_1 = \alpha_3, \dots, \alpha'_{n-2} = \alpha_n\}$  to a base  $\Delta'$  for  $\Phi$  and apply Lemma 12 to get in turn  $\dim(W_3 \cap W_4) < n-3$ ,  $\dim(W_3 \cap W_4 \cap W_5) < n-4$ ,  $\dots$ ,  $\dim(W_3 \cap \dots \cap W_n) < n - (n-1) = 1$ . But by assumption,  $v \in W_3 \cap \dots \cap W_n$ . This contradiction gives the result.  $\square$

**Corollary 14.** If  $n \geq 4$  and  $M$  is simple, then for  $\alpha \in \Phi$  the eigenspace of  $\mathcal{A}_\alpha$  are all of dimension greater than 1.

*Proof.* If  $W = Cv$  is a 1-dimensional eigenspace of  $\mathcal{A}_\alpha$ , then take  $\alpha_1 = \alpha$  in base  $\Delta$  and note that  $\mathcal{A}_{\alpha_3}, \dots, \mathcal{A}_{\alpha_n}$  all have  $v$  as an eigenvector. Lemma 13 implies that  $M$  is not simple.  $\square$

*Proof of Main Theorem.* Assume that  $M$  is simple torsion-free of degree  $k \leq n-2$  and  $n \geq 4$ . If  $k = 2$ , then we have the contradiction concerning the dimension of an eigenspace of  $\mathcal{A}_\alpha$  given to us by Lemma 12 and Corollary 14. Hence,  $k \geq 3$ .

This proves that the result is true for  $n = 4$  and begins our inductive proof. Now, assume that the result is true for all  $m$ ,  $4 \leq m < n$ , and  $n \geq 5$ .

As our first step, we establish the claim that, when  $n \geq 5$ ,  $k = n-2$  and each operator  $\mathcal{A}_\alpha$  has a unique eigenvalue whose eigenspace has dimension  $d = n-3$ . For  $n = 5$ , Lemma 12 and Corollary 14 imply that the claim is true. Continuing, we assume that  $n \geq 6$ . Let  $V$  be either an eigenspace or a generalized eigenspace of  $\mathcal{A}_\alpha$ . By Corollary 14,  $\dim V \geq 2$ . Suppose that  $2 \leq \dim V \leq n-4$ . Take a base  $\Delta = \{\alpha = \alpha_1, \dots, \alpha_n\}$ . Let  $A_m$  be the subalgebra generated by the root vectors in  $\Phi^3$ .  $U(A_m)V$  is a torsion-free  $A_m$ -module, and since  $m = n-2 \geq 4$ , the induction assumption says that there is a 1-dimensional  $U_0(A_m)$ -submodule of  $V$ , contrary to Lemma 13. If  $\mathcal{A}_\alpha$  has two distinct eigenvalues with corresponding dimensions  $d_1$  and  $d_2$ , then  $k \geq d_1 + d_2 > 2n-8$  and so we have arrived at the contradiction that  $2n-8 < k \leq n-2$  or  $n < 6$ . Hence, the claim is true.

Our second claim is that if  $n \geq 5$  and  $W_i$  is the unique eigenspace of  $\mathcal{A}_{\alpha_i}$  of dimension  $n-3$  for  $\alpha_i \in \Delta$ , then  $\dim(W_{i_1} \cap \dots \cap W_{i_j}) \geq n-2-j$ . This is known for  $j = 1$ . Inductively assume that  $\dim(W_{i_1} \cap \dots \cap W_{i_{j-1}}) \geq n-2-(j-1) = n-1-j$ . Then

$$\begin{aligned} n-2 &\geq \dim(W_{i_1} \cap \dots \cap W_{i_{j-1}} + W_{i_j}) \\ &= \dim(W_{i_1} \cap \dots \cap W_{i_{j-1}}) + \dim(W_{i_j}) - \dim(W_{i_1} \cap \dots \cap W_{i_j}) \\ &\geq n-1-j + n-3 - \dim(W_{i_1} \cap \dots \cap W_{i_j}). \end{aligned}$$

Thus  $\dim(W_{i_1} \cap \dots \cap W_{i_j}) \geq n-2-j$ .

When  $j = n-3$ ,  $\dim(W_{i_1} \cap \dots \cap W_{i_j}) \geq 1$ . If  $W_1 \cap \dots \cap W_{j-1} = W_1 \cap \dots \cap W_j$ , then  $\mathcal{A}_{\alpha_j}$  is a scalar on  $W = W_1 \cap \dots \cap W_{j-1}$  and  $M$  is not simple by Lemma 13. Therefore,  $W_1 \supset W_1 \cap W_2 \supset \dots \supset W_1 \cap \dots \cap W_{n-3} \neq \{0\}$  and so  $\dim(W_1 \cap \dots \cap W_{n-4}) = 2$  and  $\dim(W_1 \cap \dots \cap W_{n-3}) = 1$ . If, for some base  $\Delta$ ,  $\dim(W_1 \cap \dots \cap W_{n-4} \cap W_n) = 1$ , then since  $\dim(W_1 \cap \dots \cap W_{n-3}) = 1$  and  $A_{\alpha_n}$  leaves  $W_1 \cap \dots \cap W_{n-3}$  invariant, we have  $W_1 \cap \dots \cap W_{n-3} \subset W_n$  and so  $W_1 \cap \dots \cap W_{n-3} = W_1 \cap \dots \cap W_{n-4} \cap W_n$ . This implies  $W_1 \cap \dots \cap W_{n-3} \subset W_{n-2}$  and  $\mathcal{A}_{\alpha_{n-2}}$  is a scalar on  $W_1 \cap \dots \cap W_{n-3}$ , contrary to Lemma 12. Hence, we assume that for all bases  $W_1 \cap \dots \cap W_{n-4} \subset W_n$ . Consider the bases

$$\Delta' = \{\alpha_1, \dots, \alpha_{n-3}, \alpha_{n-2} + \alpha_{n-1}, -\alpha_{n-1}, \alpha_{n-1} + \alpha_n\}$$

and

$$\Delta'' = \{\alpha_1, \dots, \alpha_{n-4}, \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}, -\alpha_{n-1}, -\alpha_{n-2}, \alpha_{n-2} + \alpha_{n-1} + \alpha_n\}.$$

Since  $\mathcal{A}_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n}$  and  $\mathcal{A}_{-(\alpha_{n-1} + \alpha_n)}$  are scalars on  $W_1 \cap \dots \cap W_{n-4}$ , Lemma 8(ii) says that  $\mathcal{A}_{\alpha_{n-2}}$  is a scalar on  $W_1 \cap \dots \cap W_{n-3} \subset W_1 \cap \dots \cap W_{n-4}$ , and we again have a contradiction to Lemma 12.  $\square$

**Example.** We provide examples of simple torsion-free  $A_n$ -modules of degrees 1,  $n-1$ , and  $n$ . To this end we first observe that the algebra  $\mathfrak{gl}(n+1, \mathbb{C})$  can be identified with an operator algebra as follows. Let  $\{x_1, \dots, x_{n+1}\}$  denote a set of commuting variables, and set  $X_i$  (respectively,  $\partial_i$ ) to be multiplication (respectively, partial differentiation) by the variable  $x_i$ . Then the map which identifies the standard basis matrices  $E_{ij}$  with the operator  $X_i \partial_j$  is a Lie algebra isomorphism. The restriction of this map to the subalgebra  $A_n$  of  $n+1 \times n+1$  traceless matrices provides an operator realization of  $A_n$  and a realization of the natural representation space  $V(\omega_1) = \text{span}_{\mathbb{C}}\{x_1, \dots, x_{n+1}\}$ .

Fix  $\vec{a} = (a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1}$  where none of the components are integers, and consider the space

$$N(\vec{a}) = \text{span}_{\mathbb{C}} \left\{ x^{\vec{b}} = x_1^{b_1} \dots x_{n+1}^{b_{n+1}} \mid b_i - a_i \in \mathbb{Z}, \text{ for all } i \text{ and } \sum_{i=1}^{n+1} (b_i - a_i) = 0 \right\}.$$

Using the realization of  $A_n$  in terms of operators we easily see that  $N(\vec{a})$  is a simple torsion-free  $A_n$ -module of degree 1. In fact in [BL1], it is shown that every simple torsion-free  $A_n$ -module of degree 1 is equivalent to such a module.

We now consider the tensor product module  $M$  given by  $M = N(\vec{a} - \varepsilon_1) \otimes V(\omega_1)$ , where  $\vec{a} - \varepsilon_1 = (a_1 - 1, a_2, \dots, a_{n+1})$ . In [BBL], it is shown that the

tensor product of a torsion-free module of finite degree and a finite-dimensional module is torsion-free. Hence,  $M$  is torsion-free.

For each  $i = 1, \dots, n+1$ , we define

$$v_i = x_1^{a_1} \cdots x_i^{a_i-1} \cdots x_{n+1}^{a_{n+1}} \otimes x_i.$$

It is easily verified that the elements  $v_i$  span the  $\Theta$  weight space of  $M$  where  $\Theta(E_{ii} - E_{i+1i+1}) = a_i - a_{i+1}$ . By direct computation we observe that subspaces

$$\mathscr{W}_1 = \text{span}_{\mathbb{C}} \left\{ \sum_{i=1}^{n+1} a_i v_i \right\}, \quad \mathscr{W}_0 = \text{span}_{\mathbb{C}} \left\{ \sum_{i=1}^{n+1} c_i v_i \mid \sum_{i=1}^{n+1} c_i = 0 \right\}$$

of  $M_{\Theta}$  are invariant under  $\mathscr{A}_{\alpha}$  for all  $\alpha \in \Phi$  and hence, by Lemma 6, they are  $U_0$  submodules. This means that  $M_0 = U\mathscr{W}_0$  and  $M_1 = U\mathscr{W}_1$  are submodules of  $M$  with  $(U\mathscr{W}_i)_{\Theta} = W_i$ .  $M_1$  is isomorphic to  $N(\vec{a})$ .

Suppose that  $M_0$  contains the proper nonzero simple submodule  $V$  of degree  $k \leq n-1$ , and hence by our Main Theorem  $k = 1$  or  $k = n-1$ . Suppose that  $k = n-1$ . Then no vector of the form  $v_i - v_j$  with  $i \neq j$  is in  $V$  because such vectors generate  $\mathscr{W}_0$  and hence generate  $M_0$  as we see from

$$X_i \partial_k X_k \partial_i (v_i - v_j) = a_i(a_k + 1)(v_i - v_j) - a_k(v_i - v_k) \quad \text{for all } k \neq i, j.$$

By a dimension argument, we know that  $(v_1 - v_2) - b(v_1 - v_3) \in V$  for some  $b \in \mathbb{C}$  and, since  $-v_2 + v_3 \notin V$ , we know that  $b \neq 1$ . Computing we have

$$\begin{aligned} X_1 \partial_4 X_4 \partial_1 ((v_1 - v_2) - b(v_1 - v_3)) \\ = [a_1(a_4 + 1)(v_1 - v_2) - a_4(v_1 - v_4)] - b[a_1(a_4 + 1)(v_1 - v_3) - a_4(v_1 - v_4)], \end{aligned}$$

from which it follows that  $v_1 - v_4 \in V$ . Hence,  $k \neq n-1$  and so  $k = 1$ . This means that there is some  $v = \sum_{i=1}^{n+1} d_i v_i \in V \cap M_{\Theta}$  with not all  $d_i = 0$  for  $i \geq 3$  which is an eigenvector for every  $X_i \partial_k X_k \partial_i$ .

$$\begin{aligned} X_1 \partial_2 X_2 \partial_1 v = (d_1(a_1 - 1)(a_2 + 1) + d_1 + d_2 a_1)v_1 + (d_1 a_2 + d_2 a_1 a_2)v_2 \\ + d_3 a_1(a_2 + 1)v_3 + \cdots + d_{n+1} a_1(a_2 + 1)v_{n+1} \end{aligned}$$

tells us that the eigenvalue for this operator is  $a_1(a_2 + 1)$ , and it forces  $d_1 = sa_1$  and  $d_2 = sa_2$  for some  $s \in \mathbb{C}$ . Similar calculations show that  $d_i = sa_i$ . Therefore,  $M_0$  is a simple torsion-free  $A_n$ -module of degree  $n$  and  $M = M_1 \oplus M_0$  unless  $\sum_{i=1}^{n+1} a_i = 0$ , in which case we have a composition series  $\{0\} \subset M_1 \subset M_0 \subset M$ . When  $M_1 \subset M_0$ , the quotient  $M_0/M_1$  is a simple torsion-free  $A_n$ -module of degree  $n-1$ .

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