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PUTNAM'S INEQUALITY FOR p-HYPONORMAL OPERATORS

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ABSTRACT. The purpose of this paper is to show the following: Let $0 < p < \frac{1}{2}$. If T is a p-hyponormal operator on a Hilbert space, then

$$\|(T^*T)^p - (TT^*)^p\| \le \frac{p}{\pi} \iint_{\sigma(T)} \rho^{2p-1} d\rho\, d\theta.$$

That is, Putnam's inequality holds for p-hyponormal operators.

1. DEFINITIONS AND PRELIMINARIES

If T is hyponormal, then

$$\pi\|T^*T - TT^*\| \le \text{meas}_2(\sigma(T)).$$

This was proved in Putnam [4] and is well known as Putnam's inequality. An operator T is called p-hyponormal if $(T^*T)^p - (TT^*)^p \ge 0$. For a number $p \ge \frac{1}{2}$, Professor Xia proved that the similar inequality holds for p-hyponormal operators: If T is p-hyponormal, then

$$\|(T^*T)^p - (TT^*)^p\| \le \frac{p}{\pi} \iint_{\sigma(T)} \rho^{2p-1}\, d\rho\, d\theta.$$

For example, see Theorem XI.5.1 of [7]. In [1], Professor Aluthge introduced and studied p-hyponormal operators for $0 < p < \frac{1}{2}$. Also in [2] and [3] spectral properties of such p-hyponormal operators have been studied. In this paper we will show that Putnam's inequality holds for such p-hyponormal operators.

Let $\mathscr{H}$ be a complex separable Hilbert space and $B(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. An operator means a bounded linear operator on $\mathscr{H}$. An operator T is said to be a p-hyponormal operator if $(T^*T)^p - (TT^*)^p \ge 0$. If $p = 1$, T is called hyponormal, and if $p = \frac{1}{2}$, T is called semi-hyponormal. It is well known that a p-hyponormal operator is q-hyponormal for $q \le p$. The set of all p-hyponormal operators in $B(\mathscr{H})$ is denoted by $p - H$. Let $p - HU$ denote the set of all operators in $p - H$ with equal defect and nullity. Hence for $T \in p - HU$ we may assume that the operator U in a polar decomposition $T = U|T|$ is unitary.

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For an operator T, we denote the spectrum, the approximate point spectrum and the residual spectrum by $\sigma(T)$, $\sigma_a(T)$ and $\sigma_r(T)$, respectively. A point $z \in \mathbb{C}$ is in the joint approximate point spectrum $\sigma_{ja}(T)$ if there exists a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that $(T-z)x_n \to 0$ and $(T-z)^* x_n \to 0$ as $n \to \infty$.

Let $\mathscr{T}$ be the set of all strictly monotone increasing continuous nonnegative functions on $\mathbb{R}^+ = [0, \infty)$. Let $\mathscr{T}_0 = \{\psi \in \mathscr{T} : \psi(0) = 0\}$.

Let $T = U|T| \in B(\mathscr{H})$ and U be unitary. For $\psi \in \mathscr{T}_0$, the mapping $\tilde{\psi}$ is defined by $\tilde{\psi}(\rho e^{i\theta}) = e^{i\theta}\psi(\rho)$ and $\tilde{\psi}(T) = U\psi(|T|)$. We need the following results.

Theorem A (Lemma I.3.1 of [7]). *Let R be a set of the complex plane $\mathbb{C}$, $T(t)$ be an operator-valued function of $t \in [0, 1]$ which is continuous in the norm topology, $\tau_t, t \in [0, 1]$, be a family of bijective mapping from R onto $\tau_t(R) \subset \mathbb{C}$, and, for any fixed $z \in R$, $\tau_t(z)$ be a continuous function of $t \in [0, 1]$ such that τ_0 is the identity function. Suppose*

$$\sigma_a(T(t)) \cap \tau_t(R) = \tau_t(\sigma_a(T(0)) \cap R) \tag{A-1}$$

for all $t \in [0, 1]$. Then, for all $t \in [0, 1]$,

$$\sigma_r(T(t)) \cap \tau_t(R) = \tau_t(\sigma_r(T(0)) \cap R), \tag{A-2}$$

$$\sigma(T(t)) \cap \tau_t(R) = \tau_t(\sigma(T(0)) \cap R). \tag{A-3}$$

Let $T = U|T|$ be p-hyponormal. Since $\ker(T) \subset \ker(T^*)$ from Lemma 1 of [3], we may assume that U is an isometry. Hence one can apply p-hyponormal operators to Lemma II.3.5 of [7]. And one can easily see that the following theorem holds:

Theorem B. *Let $T = U|T|$ be p-hyponormal. Then there exists a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that U extends to a unitary operator V on $\mathscr{K}$ and $|T|$ extends to a semi-positive operator $|A|$ on $\mathscr{K}$. Also T extends to $A = V|A| \in p-HU$ and $\sigma(T) \subset \sigma(A) \subset \sigma(T) \cup \{0\}$, $\sigma(|A|^2) = \sigma(|T|^2) \cup \{0\}$ and $\|(T^*T)^p - (TT^*)^p\| = \|(A^*A)^p - (AA^*)^p\|$.*

Theorem C (Theorem 8 of [3]). *Let $T = U|T|$ be p-hyponormal. Then*

$$\sigma_a(T) = \sigma_{ja}(T).$$

Theorem D (Lemma VI.3.10 of [7]). *Let $T = U|T|$ be a semi-hyponormal operator. Then $\sigma(|T|) \subset \pi(\sigma(T))$, where π is the mapping defined by $\pi(z) = |z|$ $(z \in \mathbb{C})$.*

Theorem E (Corollary V.3.6 of [7]). *Let T be a semi-hyponormal operator. Then*

$$\|(T^*T)^{1/2} - (TT^*)^{1/2}\| \le \frac{1}{2 \cdot \pi} \iint_{\sigma(T)} d\rho\, d\theta.$$

2. Theorem

Throughout this paper, let p be $0 < p < \frac{1}{2}$.

Theorem 1. *Let $T = U|T| \in B(\mathscr{H})$, U be unitary and $\psi \in \mathscr{T}_0$. Then*

$$\sigma_{ja}(\tilde{\psi}(T)) = \tilde{\psi}(\sigma_{ja}(T)).$$

Proof. Take $z = \rho e^{i\theta} \in \sigma_{ja}(T)$ arbitrarily.

(i) In the case of $\rho > 0$, from Lemma I.2.4 of [7] there exists a sequence $\{x_n\}$ of unit vectors such that

(1) $$\lim_{n\to\infty} \|(U - e^{i\theta})x_n\| = 0,$$

(2) $$\lim_{n\to\infty} \|(|T| - \rho)x_n\| = 0.$$

For any $\varepsilon > 0$, take a polynomial $P_\varepsilon(\cdot)$ such that

$$\max_{\rho\in\sigma(|T|)} |\psi(\rho) - P_\varepsilon(\rho)| < \varepsilon.$$

Hence, $P_\varepsilon(|T|)$ converges to $\psi(|T|)$ strongly. Since $P_\varepsilon(\cdot)$ is the polynomial, it follows from (2) that

$$\lim_{n\to\infty} \|(P_\varepsilon(|T|) - P_\varepsilon(\rho))x_n\| = 0.$$

Thus we have

(3) $$\lim_{n\to\infty} \|(\psi(|T|) - \psi(\rho))x_n\| = 0.$$

Therefore, from (1) and (3)

$$\tilde{\psi}(\rho e^{i\theta}) = e^{i\theta}\psi(\rho) \in \sigma_{ja}(U\psi(|T|)) = \sigma_{ja}(\tilde{\psi}(T)).$$

Hence we have

(4) $$\tilde{\psi}(\sigma_{ja}(T)) \subset \sigma_{ja}(\tilde{\psi}(T)).$$

(ii) In the case of $\rho = 0$, let $0 \in \sigma_{ja}(T)$. Then there exists a sequence $\{y_n\}$ of unit vectors such that

(5) $$\lim_{n\to\infty} \|U|T|y_n\| = 0$$

and

(6) $$\lim_{n\to\infty} \|\,|T|U^*y_n\| = 0.$$

Since $\psi \in \mathscr{T}_0$, there exists a sequence $\{Q_m(\cdot)\}$ of polynomials such that $Q_m(0) = 0$ for every m and $Q_m \to \psi$ uniformly on $[0, \|T\|]$ as $m \to \infty$. Since $|T|y_n \to 0$ $(n \to \infty)$, by (5) it holds that $Q_m(|T|)y_n \to 0$ $(n \to \infty)$ for every m. Hence we have $\psi(|T|)y_n \to 0$ $(n \to \infty)$. And since $|T|U^*y_n \to 0$ $(n \to \infty)$, by (6) it holds that $Q_m(|T|)U^*y_n \to 0$ $(n \to \infty)$. Hence we have $\psi(|T|)U^*y_n \to 0$ $(n \to \infty)$. Therefore

$$\lim_{n\to\infty} \|U\psi(|T|)y_n\| = 0$$

and

$$\lim_{n\to\infty} \|\psi(|T|)U^*y_n\| = 0.$$

Therefore we have $0 \in \sigma_{ja}(\tilde{\psi}(T))$.

By replacing $\tilde{\psi}$ by $\tilde{\psi}^{-1}$ and T by $\tilde{\psi}(T)$ in (4), we have

$$\sigma_{ja}(\tilde{\psi}(T)) \subset \tilde{\psi}(\sigma_{ja}(T)).$$

This completes the proof.

Theorem 2. *Let* $T = U|T| \in p - HU$, $\Psi(\cdot, t) \in \mathscr{T}_0$ *for* $t \in [0, 1]$ *and, for each* $p \in \sigma(|T|)$, $\Psi(\rho, t)$ *be a continuous function at* $t \in [0, 1]$ *satisfying* $\Psi(\rho, 0) = \rho$. *Define*

$$T(t) = U\Psi(|T|, t) \quad \text{and} \quad \tau_t(\rho e^{i\theta}) = e^{i\theta}\Psi(\rho, t) \qquad (t \in [0, 1]).$$

Let R *be a set in the complex plane. Suppose*

$$\sigma_{ja}(T(t)) \cap \tau_t(R) = \sigma_a(T(t)) \cap \tau_t(R), \qquad t \in [0, 1]. \tag{7}$$

Then, for $t \in [0, 1]$,

$$\sigma_a(T(t)) \cap \tau_t(R) = \tau_t(\sigma_a(T) \cap R), \tag{8}$$

$$\sigma_r(T(t)) \cap \tau_t(R) = \tau_t(\sigma_r(T) \cap R), \tag{9}$$

and

$$\sigma(T(t)) \cap \tau_t(R) = \tau_t(\sigma(T) \cap R). \tag{10}$$

Proof. By the definition of τ_t, it is clear that $\tau_0 = \text{id}$ (identity mapping) and $\tau_t(\cdot)$ is continuous with respect to t.

Obviously, τ_t is a bijective function from $\sigma(|T|)$ to $\tau_t(\sigma(|T|))$. Since

$$\|T(t) - T(s)\| = \|\Psi(|T|, t) - \Psi(|T|, s)\|$$

and $\Psi(\rho, t)$ is continuous with respect to t, we have that $T(t)$ is continuous at $t \in [0, 1]$. And since $\tau_t(T) = U\Psi(|T|, t) = T(t)$, by Theorem 1 we have

$$\sigma_{ja}(T(t)) = \tau_t(\sigma_{ja}(T)).$$

So that, from (7),

$$\sigma_a(T(t)) \cap \tau_t(R) = \tau_t(\sigma_{ja}(T)) \cap \tau_t(R).$$

From Theorem C, $\sigma_{ja}(T) = \sigma_a(T)$. Hence (8) holds. Since τ_t is continuous and $T = T(0)$, we have

$$\sigma_a(T(t)) \cap \tau_t(R) = \tau_t(\sigma_a(T(0)) \cap R).$$

Since (A-1) of Theorem A holds, (A-2) and (A-3) hold. Therefore (9) and (10) hold. This completes the proof.

Theorem 3. *Let* $T = U|T| \in p - HU$ *and* $\psi \in \mathscr{T}_0$. *If* $\tilde{\psi}(T) \in p - HU$, *then the following hold*:

$$\sigma_a(\tilde{\psi}(T)) = \tilde{\psi}(\sigma_a(T)), \tag{11}$$

$$\sigma_r(\tilde{\psi}(T)) = \tilde{\psi}(\sigma_r(T)), \tag{12}$$

and

$$\sigma(\tilde{\psi}(T)) = \tilde{\psi}(\sigma(T)). \tag{13}$$

Proof. For $t \in [0, 1]$, set

$$\Psi(\rho, t) = [(1 - t)\rho^{2p} + t\psi(\rho)^{2p}]^{1/2p}.$$

Since $\psi \in \mathscr{T}_0$, it is clear that $\Psi(\cdot, t)$ is a strictly monotone increasing continuous function such that $\Psi(0, \cdot) = 0$ and $\Psi(\rho, \cdot) \geq 0$. Hence $\Psi(\cdot, t) \in \mathscr{T}_0$. Define $\Psi(T, t) = T(t)$ and $\tau_t(\rho e^{i\theta}) = e^{i\theta}\Psi(\rho, t)$. It follows that

$$T(t) = U[(1 - t)|T|^{2p} + t\psi(|T|)^{2p}]^{1/2p},$$

and

$$\tau_t(\rho e^{i\theta}) = e^{i\theta}[(1-t)\rho^{2p} + t\psi(\rho)^{2p}]^{1/2p}.$$

Hence

$$\begin{aligned}(T(t)^*T(t))^p &- (T(t)T(t)^*)^p \\ &= (1-t)[|T|^{2p} - U|T|^{2p}U^*] + t[\psi(|T|)^{2p} - U\psi(|T|)^{2p}U^*] \geq 0.\end{aligned}$$

Therefore, $T(t) \in p-HU$. Thus, it follows from Theorem C that

$$\sigma_{ja}(T(t)) = \sigma_a(T(t)).$$

So

$$\sigma_{ja}(T(t)) \cap \tau_t(R) = \sigma_a(T(t)) \cap \tau_t(R).$$

Here, letting $R = \mathbb{C}$ it follows that (7) of Theorem 2 holds.

Hence, (8), (9) and (10) of Theorem 2 hold. Especially, by taking $t = 1$, it follows that (11), (12) and (13) hold. This completes the proof.

Theorem 4. *Let $T = U|T| \in p-H$. If $r \in \sigma(T^*T)$, then there exists $z \in \sigma(T)$ such that $|z|^2 = r$.*

Proof. (i) In the case of $T = U|T| \in p-HU$, set $S = U|T|^{2p}$. Since

$$(S^*S)^{1/2} - (SS^*)^{1/2} = |T|^{2p} - U|T|^{2p}U^* \geq 0,$$

S is a semi-hyponormal operator. Now let $\psi(\rho) = \rho^{1/2p}$ $(\rho \geq 0)$. Then $\psi \in \mathscr{T}_0$ and

$$\tilde{\psi}(S) = U(|T|^{2p})^{1/2p} = U|T| \in p-HU. \tag{14}$$

It follows from Theorem 3 that $\sigma(\tilde{\psi}(S)) = \tilde{\psi}(\sigma(S))$. Hence by (14) we have

$$\sigma(T) = \tilde{\psi}(\sigma(S)). \tag{15}$$

Since $r \in \sigma(T^*T)$, it follows that $r^p \in \sigma(|T|^{2p}) = \sigma(|S|)$. Since $S = U|S|$ is semi-hyponormal, from Theorem D it holds that there exists $e^{i\theta} \in \sigma(U)$ such that

$$r^p e^{i\theta} \in \sigma(S) = \sigma(U|T|^{2p}).$$

Hence, by (15) and $\tilde{\psi}(r^p e^{i\theta}) = (r^p)^{1/2p} \cdot e^{i\theta} = \sqrt{r}e^{i\theta}$,

$$\sqrt{r}e^{i\theta} \in \sigma(\tilde{\psi}(S)) = \sigma(T).$$

Put $z = \sqrt{r}e^{i\theta}$. Then $z \in \sigma(T)$ and $|z|^2 = r$. So this z is a desired number.

(ii) In the case of $T \in p-H$, if $r = 0$, then it is clear that $0 \in \sigma(T)$. Hence we may assume $r \neq 0$. From Theorem B it follows that there exist a Hilbert space $\mathscr{K} \supset \mathscr{H}$ and operators A, U and $|A|$ on $\mathscr{K}$ such that V is unitary, $A = V|A|$ is a polar decomposition of A and $\sigma(T) \subset \sigma(A) \subset \sigma(T) \cup \{0\}$. Since $r \neq 0$ and $\sigma(|A|^2) = \sigma(|T|^2) \cup \{0\}$, we have $r \in \sigma(A^*A)$. Since $A = V|A| \in p-HU$, from case (i) it follows that there exists $z \in \sigma(A)$ such that $|z|^2 = r$. Since $z \neq 0$, we have $z \in \sigma(T)$ and this z is a desired number.

From (i) and (ii), the proof is complete.

Theorem 5. *If T is a p-hyponormal operator, then*

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \iint_{\sigma(T)} \rho^{2p-1}\, d\rho\, d\theta. \tag{16}$$

Proof. (i) In the case of $T \in p-HU$, put $S = U|T|^{2p}$. Then S is a semi-hyponormal operator. Hence by Theorem E it holds that

$$\|(S^*S)^{1/2} - (SS^*)^{1/2}\| \leq \frac{1}{2\cdot\pi}\iint_{\sigma(S)} dr\,d\theta. \tag{17}$$

Now we define the mapping ψ by $\psi(\rho) = \rho^{1/2p}$. Then $\psi \in \mathscr{T}_0$ and $\tilde{\psi}(S) = U\psi(|S|) = T$. From Theorem 3, we have $\sigma(\tilde{\psi}(S)) = \tilde{\psi}(\sigma(S))$. Therefore it holds that

$$\sigma(T) = \tilde{\psi}(\sigma(S)). \tag{18}$$

Since $(S^*S)^{1/2} - (SS^*)^{1/2} = (T^*T)^p - (TT^*)^p$, by the transformation $r = \rho^{2p}$, from (17) and (18) we have

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi}\iint_{\sigma(T)} \rho^{2p-1}\,d\rho\,d\theta.$$

Therefore, (16) holds.

(ii) In the case of $T \in p-H$, let A be the extended operator of T of Theorem B. Since $A \in p-HU$, from (i) it holds that

$$\|(A^*A)^p - (AA^*)^p\| \leq \frac{p}{\pi}\iint_{\sigma(A)} \rho^{2p-1}\,d\rho\,d\theta.$$

Since

$$\|(T^*T)^p - (TT^*)^p\| = \|(A^*A)^p - (AA^*)^p\|$$

and

$$\sigma(T) \subset \sigma(A) \subset \sigma(T) \cup \{0\},$$

It is easy to see that (16) holds. This completes the proof.

Corollary. *Let $T \in p-H$. If one of the following conditions holds, then T is a normal operator.*

(i) *T is a compact operator.*

(2) *$m_2(\sigma(T)) = 0$, where m_2 denotes the planar Lebesgue measure.*

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