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FILTERS AND GAMES

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ABSTRACT. We obtain game-theoretic characterizations for meagerness and rareness of filters on ω.

One of the classical methods for obtaining a set of real numbers which does not have the property of Baire is to interpret appropriate filters on $\mathbb{N} = \{1, 2, 3, \dots\}$ as subsets of $[0, 1]$. Filters which result in a set having the property of Baire have nice combinatorial characterizations, due to Talagrand [3]:

Theorem 1 (Talagrand). *For a filter $\mathcal{F}$ on ω the following are equivalent:*

(1) *$\mathcal{F}$ does not have the property of Baire.*
(2) *The set of enumeration functions of sets in $\mathcal{F}$ is unbounded in ${}^{\omega}\omega$, ordered by eventual domination.*
(3) *For every partition of ω into disjoint finite sets, there is a set in $\mathcal{F}$ which is disjoint from infinitely many blocks in the partition.*

In particular, we see that a filter is meager if, and only if, it has the property of Baire. We use this without further notice below.

These combinatorial characterizations suggest certain infinite two-player games. We study such a game in section 1. We prove that having the property of Baire is equivalent to the assertion that player ONE of our game has a winning strategy (Theorem 3). In the second section we study a natural variation of the first game and show that here, being non-rare is equivalent to ONE having a winning strategy in this game (Theorem 5).

Our notation is mostly standard; the only exception may be that we use $\frown$ to denote concatenation of sequences. From now on, let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ be a non-principal filter.

1. THE GAME $\mathfrak{G}_1(\mathcal{F})$

This game is played as follows: In the k-th inning, player ONE chooses $m_k \in \mathbb{N}$ and player TWO responds with $n_k \in \mathbb{N}$. Player TWO wins the play

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$(m_1, n_1, \dots, m_k, n_k, \dots)$ if:

(1) $n_1 < n_2 < \cdots < n_k < \cdots$,
(2) there are infinitely many k such that $m_k < n_k$, and
(3) $\{n_1, n_2, \dots, n_k, \dots\} \in \mathcal{F}$.

Otherwise, ONE wins.

Theorem 2. *TWO does not have a winning strategy in* $\mathfrak{G}_1(\mathcal{F})$.

Proof. Suppose on the contrary that F is a winning strategy for TWO in $\mathfrak{G}_1(\mathcal{F})$. Here, and below, we may assume without loss of generality that F depends on only ONE's moves—TWO's moves can be decoded from these, using F.

Claim: For each finite sequence σ of natural numbers, there exists a finite sequence τ of natural numbers and a natural number n such that $F(\sigma \frown \tau \frown (y)) > y$ for each $y > n$.

Proof of claim. If the claim were false, then there would be a finite sequence σ_0 such that for every finite sequence τ and every natural number n, there is a $y > n$ such that $F(\sigma_0 \frown \tau \frown (y)) \leq y$. Fix σ_0 and choose $y_1 < y_2 < y_n < \cdots$ so that $\max(\sigma_0) < y_1$ and $F(\sigma_0 \frown (y_1, \dots, y_n) \frown (y_{n+1})) \leq y_{n+1}$ for each n. Then TWO lost the play where ONE begins the game by picking the values of σ_0 first and then the numbers y_n consecutively. This contradicts our hypothesis that F is a winning strategy for TWO.

Using the claim, we now derive a contradiction from the hypothesis that F is a winning strategy for TWO. To begin, let σ be the empty sequence. Pick $\tau_\varnothing$ as in the claim. Then choose $y_0 > \max \tau_\varnothing$ also as in the claim. Put $\sigma_0 = \tau_\varnothing \frown (y_0)$ and pick τ_0 as in the claim.

Next choose $y_1 > \max\{F(\sigma_0 \frown \tau_0 \restriction_j) : j < \omega\}$ and put $\sigma_1 = \tau_\varnothing \frown (y_1)$; choose τ_1 as in the claim.

Next choose $y_2 > \max\{F(\sigma_1 \frown \tau_1 \restriction_j) : j < \omega\}$ and put $\sigma_2 = \sigma_0 \frown \tau_0 \frown (y_2)$ and then choose τ_2 as in the claim.

Then choose $y_3 > \max\{F(\sigma_2 \frown \tau_2 \restriction_j) : j < \omega\}$ and put $\sigma_3 = \sigma_1 \frown \tau_1 \frown (y_3)$; then choose τ_3 as in the claim.

Continuing like this we choose three sequences

$$(y_0, y_1, y_2, \dots), \quad (\sigma_0, \sigma_1, \sigma_2, \dots), \quad (\tau_0, \tau_1, \tau_2, \dots)$$

such that

(1) $\sigma_{2i+1} = \sigma_{2i-1} \frown \tau_{2i-1} \frown (y_{2i+1})$,
(2) $\sigma_{2i+2} = \sigma_{2i} \frown \tau_{2i} \frown (y_{2i+2})$,
(3) τ_{2i+1} is chosen as in the claim for σ_{2i+1},
(4) τ_{2i} is chosen as in the claim for σ_{2i},
(5) y_{2i+1} is chosen as in the claim for $\sigma_{2i-1} \frown \tau_{2i-1}$, so that $y_{2i+1} > \max\{F(\sigma_{2i} \restriction_j) : j < \omega\}$, and
(6) y_{2i+2} is chosen as in the claim for $\sigma_{2i} \frown \tau_{2i}$, so that

$$y_{2i+2} > \max\{F(\sigma_{2i+1} \restriction_j) : j < \omega\}.$$

Let f and g be the unique sequences of natural numbers such that $\sigma_{2i} \subset f$ and $\sigma_{2i+1} \subset g$ for all i. Then by construction the sets

$$A_f = \{F(F \restriction_n) : n \in \mathbb{N}\} \quad \text{and} \quad A_g = \{F(g \restriction_n) : n \in \mathbb{N}\}$$

have finite intersection, and both are response sets for TWO using the strategy F. But then at least one of these sets is not in the non-principal filter $\mathscr{F}$! Thus there is a play for ONE against F which defeats TWO. $\square$

Theorem 3. *The following statements are equivalent*:

(1) $\mathscr{F}$ *is a meager filter.*
(2) *ONE has a winning strategy in* $\mathfrak{G}_1(\mathscr{F})$.
(3) $\mathfrak{G}_1(F)$ *is determined.*

Proof. The implication $1 \Rightarrow 2$ follows from the negation of (2) in Talagrand's theorem. The implication $2 \Rightarrow 3$ is trivial. We show that $(3) \Rightarrow (1)$.

We already know that TWO does not have a winning strategy. Let F be a winning strategy for ONE in $\mathfrak{G}_1(\mathscr{F})$. We may assume that $\max\{x_1, \dots, x_n\} + 1 \leq F(x_1, \dots x_n)$ for all $x_1 < \cdots < x_n$. Define a function f so that $f(0) = F(\varnothing)$ and for $n > 1$,

$$f(n) = \max\{F(t_1, t_2, \dots, t_j) : t_i \leq n \quad \text{and} \quad t_1 < \cdots < t_j \text{ for } i \leq j \leq n\}.$$

Then f is monotonic—i.e., $f(m) \leq f(n)$ whenever $m \leq n$. For each n define $h_n(m) = f^m(n)$ for each m. Then choose g so that

$$g(k) = \max\{h_m(n) : m, n \leq k\} + 1 \quad \text{for all } k.$$

We claim that if $X \in \mathscr{F}$, then g dominates $enum_X$, the enumueration function of X. Fix such an X and write $X = \{x_1, x_2, \dots\}$ in increasing order. Since F is a winning strategy for ONE, the play

$$F(\varnothing), x_1, F(x_1), x_2, F(x_1, x_2), x_3, F(x_1, x_2 x_3), \dots$$

is won by ONE. Thus, there is an $N \in \mathbb{N}$ such that $x_{n+1} \leq F(x_1, \dots, x_n)$ for $n \geq N$.

We see in particular that:

$$x_{N+1} \leq F(x_1, \dots, x_N) \leq f(x_N),$$
$$x_{N+2} \leq F(x_1, \dots, x_{N+1}) \leq f(x_{N+1}) \leq f^2(x_N),$$

and in general,

$$x_{N+k} \leq F(x_1, \dots, x_{N+k-1}) \leq f(x_{N+k-1}) \leq f^k(x_N) = h_{x_N}(k)$$

for each k.

But g eventually dominates h_{x_N}. Choose $K > x_N$ so large that $h_{x_N}(k) < g(k)$ for all $k \geq K$. Then $x_{N+k} < g(k)$ for all $k \geq K$; in particular, $x_k < g(k)$ for all $k \geq K$.

We have shown that the set of enumeration functions of elements of $\mathscr{F}$ is bounded; by Talagrand's theorem, $\mathscr{F}$ is meager. $\square$

2. The game $\mathfrak{G}_2(\mathscr{F})$

In the game $\mathfrak{G}_1(\mathscr{F})$, TWO's objective was to play a sequence (enumerating a set from the filter) not eventually dominated by ONE's sequence. What is the situation when we change TWO's objective to playing a sequence (enumerating a set from the filter) which actually eventually dominates ONE's sequence? We consider this now: In the k-th inning, player ONE chooses $m_k \in \mathbb{N}$ and player TWO responds with $n_k \in \mathbb{N}$. TWO wins the play $(m_1, n_1, \dots, m_k, n_k, \dots)$

if:

(1) $(n_1, \dots, n_k, \dots)$ eventually dominates $(m_1, m_2, \dots, m_k, \dots)$ *and*
(2) $\{n_1, n_2, \dots, n_k, \dots\} \in \mathscr{F}$.

Otherwise, ONE wins.

Theorem 4. *TWO does not have a winning strategy in* $\mathfrak{G}_2(\mathscr{F})$.

Proof. This theorem follows immediately from Theorem 2, because the game $\mathfrak{G}_2(\mathscr{F})$ is harder for TWO than $\mathfrak{G}_1(\mathscr{F})$. However, there is a much simpler argument than that for Theorem 2.

Suppose on the contrary that F is a winning strategy for TWO. Now consider the game where player ONE starts the game by making an arbitrary move, and then also uses TWO's strategy F, while TWO uses TWO's strategy to respond to ONE.

Consider the play $(m_1, n_1, m_2, n_2, \dots, m_k, n_k, \dots)$ where m_i is ONE's i-th move, n_i is TWO's i-th move and $n_i = F(m_1, \dots, m_i)$ and $m_{i+1} = F(n_1, \dots, n_i)$, and $m_{i+1} > n_i > m_i$ for all i.

Since F is a winning strategy, we have

(1) $\{m_j : j \in \mathbb{N}\} \in \mathscr{F}$ and $\{n_j : j \in \mathbb{N}\} \in \mathscr{F}$ and
(2) there is an l such that $m_j < n_j$ for all $j > l$.

But then $\{m_j : j \in \mathbb{N}\} \cap \{n_j : j \in \mathbb{N}\}$ is finite, contradicting the fact that both sets are from the non-principal filter $\mathscr{F}$. □

Definition 1. $\mathscr{F}$ is a rare filter if there is for each partition $\{I_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into disjoint finite sets, an $X \in \mathscr{F}$ such that $|X \cap I_n| \leq 1$ for each n.

Theorem 5. *The following statements are equivalent*:

(1) *ONE has a winning strategy in* $\mathfrak{G}_2(\mathscr{F})$.
(2) $\mathfrak{G}_2(\mathscr{F})$ *is determined.*
(3) $\mathscr{F}$ *is not rare.*

Proof. The implication $(1) \Rightarrow (2)$ is trivial.

For the implication that (2) implies (3), assume that $\mathscr{F}$ is rare. Since TWO does not have a winning strategy in this game, we consider strategies for ONE only. We show that ONE does not have a winning strategy.

Consider a strategy F for ONE. We may assume that for all $x_1 < \cdots < x_n$ the strategy F satisfies

$$\max\{x_1, \dots, x_n\} + 1 < F(x_1, \dots, x_n)$$

and

$$F(x_1) < F(x_1, x_2) < \cdots < F(x_1, \dots, x_n).$$

Define g by $g(1) = F(\varnothing)$, and $g(n+1) = \max\{F(j_1, \dots, j_i) : j_1 < \cdots < j_i \leq n+1\} + g(n)$ for each $n \in \mathbb{N}$. Observe that if $m < n$, then $g(m) < g(n)$.

Put $h(n+1) = g(h(n))$ for each n, and $h(1) = F(\varnothing) + 1$. Then h is strictly increasing. Consider the partition $\{I_n : n \in \mathbb{N}\}$ where $I_n = [h(n), h(n+1))$ for $n > 1$ and $I_1 = [1, h(1))$.

Since $\mathscr{F}$ is rare, choose an $X \in \mathscr{F}$ such that $|X \cap I_n| \leq 1$ for each n. Enumerate X in increasing order as $\{x_n : n \in \mathbb{N}\}$. Then choose an infinite subset Y of X such that $X \setminus Y \in \mathscr{F}$ (for example, let Y be the complement of a selector from $\mathscr{F}$ of the partition $K_1, K_2, \dots$ where $K_1 = [1, x_1)$ and $K_{n+1} =$

$[x_{n^2}, x_{(n+1)^2}))$. Enumerate Y increasingly as $\{y_1, \dots, y_n, \dots\}$. We may assume that $1 < y_1$. Put $J_1 = [1, y_1)$, and for all n put $J_{n+1} = [y_n, y_{n+1})$. Since $\mathscr{F}$ is rare, we find a $Z \in \mathscr{F}$ such that $|Z \cap J_n| \leq 1$ for each n. Put $T = X \cap Z$. Then T is also a selector of the family $J_1, J_2, \dots, J_n, \dots$, is in $\mathscr{F}$, and contains no endpoint of any of the J_n's.

We now have three sequences: $(n_1, n_2, \dots, n_k, \dots)$, $(x_{m_1}, x_{m_2}, \dots, x_{m_k}, \dots)$, and $(y_{s_1}, y_{s_2}, \dots, y_{s_k}, \dots)$, such that:

(1) $x_{m_i} \in I_{n_i} \cap J_{s_i}$ for each i, and
(2) $\max(I_{n_i}) < \max(J_{s_i}) < \min(I_{n_{i+1}})$ for each i.

But then we have for each i the inequalities

$$h(n_i) \leq x_{m_i} < h(n_i + 1) < y_{s_i} < h(n_{i+1}).$$

We claim that TWO wins the play against F where TWO plays $x_{m_1}, x_{m_2}, x_{m_3}, \dots$. To see this, first observe that $F(\varnothing) < h(1) \leq h(n_1) \leq x_{m_1} < h(n_1 + 1) < y_{s_1} < h(n_2)$; thus, $F(x_{m_1}) < g(x_{M_1}) < g(h(n_1 + 1)) < h(n_1 + 2) \leq h(n_2) \leq x_{n_2}$. Again applying the inequalities above and the definition of h, we see that $F(x_{m_1}, x_{m_2}) < x_{m_3}$, and so on.

Thus the play $(F(\varnothing), x_{m_1}, F(x_{m_1}), x_{m_2}, F(x_{m_1}, x_{m_2}), x_{m_3}, \dots)$ has the properties that the set of moves by TWO is in $\mathscr{F}$, and as a sequence eventually dominates the sequence of moves by ONE. It follows that F is not a winning strategy of ONE. This completes the proof of $\neg(3) \Rightarrow \neg(2)$.

Next we show that (3) implies (1): Let $\mathscr{F}$ be a non-rare filter, and choose a partition $\{I_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ into disjoint finite sets such that each element of $\mathscr{F}$ meets infinitely many of the I_n's in more than one point. Player ONE's strategy F will be a simple 1-tactic (i.e., it depends only on the most recent move of the opponent). Define F as follows: Let k be given. Fix n so that $k \in I_n$ and put $F(k) = (\max \bigcup_{j \leq n+k} I_j) + n + k + 1$.

To see that F is winning for ONE, suppose that $(m_1, n_1, m_2, n_2, \dots, m_k, n_k, \dots)$ is an F-play. Then $m_{k+1} = F(n_k)$ for each k. For each k let j_k be such that $n_k \in I_{j_k}$. By the definition of F we see that if for all but finitely many k we have $m_k < n_k$, then for all but finitely many k we have $m_{k+1} < j_{k+1} < m_{k+2} < j_{k+2}$; thus $X = \{n_k : k \in \mathbb{N}\}$ meets all but finitely many I_n in at most one point. But then $X \notin \mathscr{F}$.

Thus, either TWO's sequence of moves enumerates a set in the filter but does not eventually dominate ONE's sequence of moves, or else TWO's sequence of moves eventually dominates ONE's sequence of moves, but does not enumerate an element of $\mathscr{F}$. In either event ONE wins. □

3. Remarks

Both of our games can be coded as Gale-Stewart games in such a way that if the filter is projective, then the corresponding Gale-Stewart game is projective. Assume the Axiom of Dependent Choices. Determinacy hypotheses such as the Axiom of Determinacy imply that our games are determined, and thus that all filters are meager and non-rare. (Projective Determinacy would imply that our games are determined for all projective filters, and thus that all projective filters are meager (hence non-rare).) But a much weaker hypothesis, namely that every set of reals has the property of Baire, already implies the determinacy of our games. In the case of rare filters this follows because Talagrand's theorem implies that these do not have the property of Baire.

In ZFC, determinacy of the games $\mathfrak{G}_2(\mathscr{F})$ is weaker than determinacy of $\mathfrak{G}_1(\mathscr{F})$. This can be seen as follows: In Theorem 5.1 of his paper [2], Kunen shows that in the random real model there are no rare ultrafilters—and thus no rare filters. In this model $\mathfrak{G}_2(\mathscr{F})$ is determined. But there are always non-Baire filters, since every non-principal ultrafilter is like that.

It is well known that in the presence of the Continuum Hypothesis or Martin's Axiom, there are rare filters and thus undetermined instances of our games. It can also be insured that these undetermined filters are (are not) P-filters, and so on.

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