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EMBEDDING THEOREMS FOR RESIDUALLY ČERNIKOV CC-GROUPS

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ABSTRACT. Embedding theorems for residually Černikov CC-groups are obtained, extending the corresponding results on FC-groups and improving some previous results on CC-groups.

1. INTRODUCTION

Groups with Černikov conjugacy classes, or CC-groups, were introduced by Polovickii [10, 11] as an extension of the concept of FC-groups. A group G is said to be a CC-group if $G/C_G(x^G)$ is a Černikov group for each $x \in G$. In the theory of FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups with some additional properties as subgroups of direct products of finite groups. Since then, his work on periodic FC-groups has been continued in a sequence of papers, such as, for example, those of Gorčakov and Tomkinson (see [15] for a complete account of this subject). The main result in this line is the characterization of the periodic residually finite FC-groups as subgroups of centrally restricted products of finite groups.

The aim of this paper is to study the natural extension of embedding theory from FC-groups to CC-groups. There have been a few papers previously written on this subject. In [10], the first of these papers, the following result is presented: a countable periodic residually Černikov CC-group is a subgroup of a direct product of Černikov groups. In [1] Franciosi, de Giovanni and Tomkinson showed that a CC-group with trivial center (and so residually Černikov) is a subgroup of a direct product of Černikov groups. We improve this result in Theorem 5.7, where we obtain the same conclusion if the residually Černikov CC-group has countable center. Finally, in [4] it is proved that a countable periodic CC-group is a section of a direct product of CC-groups. Here (Theorem 4.2) we obtain an analogous result for periodic residually Černikov CC-groups of arbitrary cardinal. In Section 5, we obtain, mainly, embedding results for $G/Z(G)$ and G' and we prove (Theorem 5.7) that a periodic residually Černikov CC-group G with G', G/G' or $Z(G)$ countable is a subgroup of a direct product of Černikov groups.

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In the following, we shall use Polovickiĭ's theorem characterization of CC-groups (Theorem 4.36 of [12]), which assures that, if G is a CC-group, then the normal closure x^G is Černikov-by-cyclic and $[G, x]$ is Černikov for every $x \in G$. Our group-theoretic notation is standard and is taken from [12] and [15]. We will refer by $\mathscr{SD}(C \cup A_0)$ ($\mathscr{QSD}(C \cup A_0)$, resp.) to the class of (quotients of, resp.) subgroups of direct products of Černikov and torsion-free abelian groups. We extend Tomkinson's definition of centrally restricted product of finite groups (see [15], p. 29) to the centrally complete product of Černikov groups, denoted by $Z_r^* C$, which is the subgroup of the cartesian product where every element has a finite number of noncentral components. Its torsion subgroup is, precisely, the centrally restricted product, denoted by $Z_r C$. A *residual system of Černikov groups* is a set of normal subgroups N_i of G with trivial intersection and such that G/N_i is a Černikov group for all $i \in I$. We denote it by $\{N_i : i \in I\}$.

2. Auxiliary results

In this section, we shall state some auxiliary results necessary for the following sections.

Lemma 2.1. (i) *The classes of FC-groups and CC-groups are closed under the formations of centrally restricted or complete products.*

(ii) *Every abelian group is an Zr^*C-group.*

(iii) $\mathscr{SD}(C \cup A_0) \leq Zr^*C \leq \mathscr{QSD}(C \cup A_0)$.

(iv) $\mathscr{SDC} \leq ZrC \leq \mathscr{QSD}C$.

Proof. The proof is an immediate consequence of the definitions. For (ii) and (iii), note that the additive group of the rational numbers is in the class Zr^*C, being a direct summand of the cartesian product of countably many copies of C_{p^∞}. □

Now, we are embedding a particularly simple class of groups, which contains the abelian groups.

Proposition 2.2. *If G is a central-by-Černikov group, $G \in \mathscr{SD}(C \cup A_0)$. Furthermore, if G is periodic, $G \in \mathscr{SDC}$.*

Proof. Let $Z = Z(G)$ so that G/Z is Černikov. It is easy to check that any abelian group is residually Černikov, and so Z is residually Černikov. Let $\{Z_i : i \in I\}$ be a Černikov residual system of Z. Each Z_i is a normal subgroup of G, and since G/Z and Z/Z_i are Černikov groups, so is G/Z_i. Therefore G is residually Černikov. Now, by Theorem 4.11 of [12], G' is Černikov. It is easy to see that there exists a normal subgroup N of G such that G/N is a Černikov group and $N \cap G' = 1$. Thus $G \leq (G/N) \times (G/G')$. Since G/G' is an abelian group, $G/G' \in \mathscr{SD}(C \cup A_0)$, and so $G \in \mathscr{SD}(C \cup A_0)$. □

The following result relates embeddings of certain subgroups with embeddings of the whole group.

Proposition 2.3. *If H is a subgroup of the CC-group G such that $G = HZ$, with $Z = Z(G)$ one has*

(i) *H is residually Černikov if and only if G is residually Černikov;*

(ii) *$H \in Zr^*C$ if and only if $G \in Zr^*C$;*

(iii) *if G is periodic, $H \in ZrC$ if an only if $G \in ZrC$.*

Proof. Let us observe that the converses are evident, and that we can deduce (iii) from (ii). In order to prove (i), let $\{H_i : i \in I\}$ be a Černikov residual system of H. Since $G = HZ$, H_i is a normal subgroup of G. Thus, $G \leq \prod\{G/Z_i : i \in I\}$. Since H/H_i is Černikov and ZH_i/H_i is central in G/H_i, we can deduce that G/H_i is central by Černikov. By Proposition 2.2, G/H_i is residually Černikov, and therefore so is G. In order to prove (ii), let us assume that $H \in Zr^*C$. We deduce that there exists a Černikov residual system $\{H_i : i \in I\}$ of H such that $H \leq Zr^*(H/H_i)$. Since $G = HZ$, each H_i is a normal subgroup of G, and we can embed G into $\prod\{G/Z_i : i \in I\}$. It is easy to see that $G \leq Zr^*(G/H_i)$. As in (i) G/H_i is a central-by-Černikov group, and by Lemma 2.1 and Theorem 2.2, $G/H_i \in Zr^*C$, and so $G \in Zr^*C$. □

The next result shows that the periodicity is not an important hypothesis when considering centrally restricted products of Černikov groups.

Proposition 2.4. *If $G \in Zr^*C$, then G is isomorphic to a subgroup of the direct product of a ZrC-group and a torsion-free abelian group.*

Proof. By hypothesis, $G \leq Zr^*\{G_i : i \in I\}$, where G_i are Černikov groups. $Zr^*G_i = ZD$, with $Z = \prod\{Z(G_i) : i \in I\}$ and $D = Dr\{G_i : i \in I\}$. It is clear that we can assume $G = ZD$. Thus $Z = Z(G)$ and G/Z is a periodic group. Let V be a maximal torsion-free subgroup of Z. Then Z/V is a periodic group, $V \cap D = 1$ and $G \leq (G/V) \times (G/D)$. The abelian group $G/D = ZD/D$ can be embedded into a torsion-free abelian group and a periodic abelian group, and the latter belongs to ZrC by Lemma 2.1. Since $G/V = (Z/V)(DV/V)$, it is a central extension of the ZrC-group $DV/V \cong D$. By Proposition 2.3, $G/V \in ZrC$, and the result follows. □

The next result shows that there exist some aspects in the theory of embeddings of CC-groups that have a better behaviour than in the FC-case. It is known that the torsion subgroup of the abelian group $\prod\{C_{p^n} : n \in N\}$ is not a subgroup of a direct product of finite groups (Example 2.6 of [15]). This is an example of a centrally restricted product of a countable number of finite groups which does not belong to the class $\mathscr{SD}F$. The next theorem shows, however, that an analogous statement is true for CC-groups, though the problem is still open for an uncountable set of indices.

Theorem 2.5. *If $G \leq Zr^*\{G_n : n \in N\}$, with G_n Černikov, then*

$$G \in \mathscr{SD}(C \cup A_0).$$

Proof. Let us suppose first that G is periodic. Let $T = T(\prod Z(G_n))$ and $D = DrG_n$ such that $G \leq ZrG_n = TD$. We can assume $G = TD$. If T is countable, so is G, and by Theorem 6 of [10] $G \in \mathscr{SD}C$. So let us assume that T is uncountable. T is abelian and periodic, so we can suppose $T \leq Dr\{E_i : i \in I\}$, where E_i are Černikov groups and I is uncountable. Since D is countable, so is $D \cap T$, and there exists a countable subset J of I such that $D \cap T \leq Dr\{E_j : j \in J\}$. Let $L = T \cap (Dr\{E_i : i \in I \backslash J\})$. Thus, T/L is a countable group. Furthermore, $L \leq G$ because $L \leq T \leq Z(G)$ and $L \cap D = L \cap T \cap D = 1$. Then $G \leq (G/L) \times (G/D)$. But G/D is a periodic abelian group, and so $G/D \in \mathscr{SD}C$. On the other hand, $G/L = (T/L)(DL/L)$. Since $DL/L \cong D$,

DL/L is a residually Černikov group, and by Proposition 2.3, so is G/L. But this group is countable and periodic, and so $G/L \in \mathscr{SD}C$, and the theorem is proved if G is periodic. In the general case, we can suppose $G = ZD$, where $Z = \prod Z(G_n)$. Proceeding as in Proposition 2.4 and keeping the same notation, we obtain $G \leq (G/V) \times (G/D)$, where G/D is abelian and G/V is a periodic group. Besides, $G/V = (Z/V)(DV/V)$. Proceeding as in the proof of Proposition 2.3, $G/V \leq Zr^*\{F_n : n \in N\}$, where F_n are periodic central-by-Černikov groups. The proof of Proposition 2.2 shows that $F_n \leq C_n \times A_n$, where C_n is Černikov and A_n is an abelian group. So $G \leq ZrC_n \times \prod A_n \times G/D$. By the first part of the proof $ZrC_n \in \mathscr{SD}C$, and so $G \in \mathscr{SD}(C \cup A_0)$. □

The next result represents a crucial point in establishing the general embedding results. It extends from FC-groups, but the proof becomes more complicated and tedious by changing finite to Černikov.

Theorem 2.6. *Let ρ be an ordinal limit. Let us assume that $\{N_\alpha : \alpha < \rho\}$ is a family of normal subgroups of the CC-group G such that $\bigcap\{N_\alpha : \alpha < \rho\} = 1$, and let us call C_α to $C_G(G/N_\alpha)$. Let $\{H_\alpha : \alpha < \rho\}$ be an ascending chain of normal subgroups of G satisfying the following properties:*

(i) $[G, H_\alpha] \leq N_\beta$, *for all* $\beta \geq \alpha$.
(ii) $G = C_\alpha H_{\alpha+2}$, *for all* $\alpha < \rho$.

Then $G \leq Zr^*\{G/N_\alpha : \alpha < \rho\}$.

Proof. Since $\{N_\alpha : \alpha < \rho\} = 1$, we can suppose that $g \leq \prod\{G/N_\alpha : \alpha < \rho\}$. Let us assume that the theorem is false. Then there must exist an element $x \in g$ with an infinite number of noncentral components. So, we can take an infinite number of ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ such that x does not belong to C_{α_n} for every n. Since, by condition (ii), $[G, x]N_\alpha = [H_{\alpha+2}, x]N_\alpha$, it follows that $[H_{\alpha_n+2}, x]$ is not contained in N_{α_n}, for each n. Now, by condition (i), $[H_{\alpha_n+2}, x] \leq N_{\alpha_m}$ for all $m \geq n+2$, because it is clear that $\alpha_{n+2} \geq \alpha_n + 2$ for all n. Let us define $M_n = \bigcap\{N_{\alpha_i} : i \geq n\}$. Thus we have an ascending chain $M_1 \leq M_2 \leq \cdots \leq M_n \leq \cdots$ such that $[H_{\alpha_n+2}, x] \leq [G, x] \cap M_{n+2}$ but $[H_{\alpha_n+2}, x]$ is not contained in M_n. So, $[G, x] \cap M_2 < [G, x] \cap M_4 < \cdots < [G, x] \cap M_{2n} < \cdots$ is a strict ascending chain in $[G, x]$. Since $[G, x]$ is a Černikov group, there must exist an m such that for $n \geq m$ the quotient $([G, x] \cap M_{2n+2})/([G, x] \cap M_{2n})$ is finite. Since $[H_{\alpha_{2n}+2}, x] \leq [G, x] \cap M_{2n+2}$, $[H_{\alpha_{2n}+2}, x]/[H_{\alpha_{2n}+2}, x] \cap M_{2n}$ is also finite. Now, $M_{2n} \leq N_{\alpha_{2n}}$, and thus, we deduce that $[H_{\alpha_{2n}+2}, x]/(H_{\alpha_{2n}+2}, x] \cap N_{\alpha_{2n}})$ is finite. This group is isomorphic to $[G, x]N_{\alpha_{2n}}/N_{\alpha_{2n}}$, and so the factors $[G, x]N_{\alpha_k}/N_{\alpha_k}$ are finite for $k = 2m, 2m+2, \ldots$. Let $\overline{G}$ be $G/(\bigcap\{N_{\alpha_k} : k = 2m, 2m+2, \ldots\})$, and let us denote by $\overline{C}$ the image of any subset C of G under the canonical map. Thus $\overline{[G, x]} \leq \prod\{[G, x]N_{\alpha_k}/N_{\alpha_k} : k \in N\}$. Since the factors of this cartesian product are finite, it follows that $\overline{[G, x]}$ is residually finite. Since $\overline{[G, x]}$ is Černikov, $\overline{[G, x]}$ must be finite. But for each natural number k, there exists $g_k \in H_{\alpha_k+2}$ such that $[g_k, x]$ does not belong to N_{α_k}. Since $[g_k, x] \in \{\bigcap N_{\alpha_j} : j > k\}$, we conclude that the elements $\overline{[g_k, x]}$ are all different. This implies that $\overline{[G, x]}$ is infinite, which is a contradiction, and the result follows. □

If G is an FC-group and N is a normal subgroup of G such that G/N is a finite group, there exists a finite subset X of G such that $G = X^G N$. This

is not true if G/N is a Černikov factor of a CC-group (see, for example, a Prufer p-group). The next lemma is the solution of this problem that we need for our purposes.

Lemma 2.7. *Let N be a normal subgroup of a CC-group G such that G/N is a Černikov group. If $C/N = Z(G/N)$, then there exists a finite subset X of G such that $G = CX^G$. (Observe that $C = C_G(G/N)$).*

Proof. Let D/N be the radicable part of G/N. Thus, G/N contains a finite subgroup S/N such that $G = SD$. There exists a finite subset X of G such that $S = \langle X \rangle N$, and therefore $G = \langle X \rangle D$. Since D/N is radicable and S/N is finite, by Lemma 3.29.1 of [12], $D/N = ([D, S]N/N)(C_{D/N}(S/N)) \leq [D, S]C/N$. Thus $G = \langle X \rangle D = SC[D, S]$. But $[D, S] \leq NX^G$, and so $G = CX^G$, and the proof is complete. □

3. Residually Černikov CC-groups with $G/Z(G)$ countable

Analogous to the classification of periodic residually finite FC-groups (cf. [5], [14], [15]) as subgroups of centrally restricted products of finite groups, we try to classify the residually Černikov CC-groups. The first step was done by Polovickii [10], who showed that a periodic residually Černikov group which is countable is a subgroup of a direct product of Černikov groups. In this section, we generalize this result to residually Černikov groups with $G/Z(G)$ countable.

Theorem 3.1. *If G is a residually Černikov CC-group with a countable residual system, then $G \in Zr^*C$. Furthermore, if G is periodic, then $G \in Z_rC$.*

Proof. Clearly, the second statement follows from the first. We know that $G \leq \prod\{F_n : n \in N\}$, where F_n is a Černikov group for each n. For each $k \geq 1$ let $G_k := G \cap (\prod\{F_n : n > k\})$. We construct by induction two chains of normal subgroups of G, $\{H_n : n \in N\}$ and $\{M_n : n \in N\}$, satisfying the following conditions: (a) $\{H_n : n \in N\}$ is an ascending chain, and H_n is the normal closure in G of a finite subset of G. (b) For each $n \geq 1$, $M_n = G_{s_n}$, where $s_n \geq n$ and $s_1 < s_2 < \cdots < s_n$. In particular, $\{M_n : n \in N\}$ is a descending chain. (c) For every $n > 1$, $G = H_nC_G(G/M_{n-1})$. (d) For every $n > 1$, $T(H_n) \cap M_n = 1$. Let us define $H_1 = 1$, $M_1 = G_1$, and let us suppose that we have constructed $n-1$ elements of both chains: $H_1 \leq H_2 \leq \cdots \leq H_{n-1}$ and $M_1 \geq M_2 \geq \cdots \geq M_{n-1}$. G/M_{n-1} is clearly a Černikov group, and so by Lemma 2.7, there exists a finite subset Y of G such that $G = Y^GC_G(G/M_{n-1})$. By hypothesis, $H_{n-1} = X^G$, for a finite subset X of G. If we define $H_n := (X \cup Y)^G$, it is clear that (a) and (c) are satisfied. $T(H_n)$ is a Černikov group, and since $\{G_n : n \in \mathbb{N}\}$ is a descending chain, there exists $m \geq 1$ such that $T(H_n) \cap G_m$ is minimal. Thus $T(H_n) \cap G_m = 1$. Let us define $s_n = \max\{m, n, s_{n-1} + 1\}$ and $m_n = G_{s_n}$. Then, it is clear that conditions (b) and (d) hold, and our construction is complete. From (b) we have $\bigcap\{M_n : n \geq 1\} = \bigcap\{G_n : n \geq 1\} = 1$. Let us define $N_0 = N_1$ and $N_i = M_{i+1}T(H_i)$, for $i \geq 1$. By Lemma 2.20 of [15], $\bigcap\{N_i : i \geq 0\} = 1$. For each $k \in N$, $[G, H_k] \leq T(H_k)$ and $T(H_k) \leq T(H_r)$ if $r \geq k$. Thus $[G, H_k] \leq N_r$, $r \geq k$. Besides, by (c) $H_{k+2}C_G(G/N_k) \geq H_{k+2}C_G(G/M_{k+1}) = G$. By Theorem 2.6 G is isomorphic to a subgroup of the centrally complete product of the G/N_i.

Since G/N_i is a quotient of $G/M_{i+1} = G/G_{s_{i+1}}$, G/N_i is Černikov, and the result follows. □

The following result extends Polovickii's theorem given in [10].

Corollary 3.2. *Let G be a residually Černikov CC-group with $G/Z(G)$ countable. Then, $G \in \mathscr{SD}(C \cup A_0)$. Furthermore, if G is periodic, $G \in \mathscr{SD}C$.*

Proof. The second sentence follows from the first. By hypothesis $G = HZ$, where H is normal in G and countable, and $Z = Z(G)$. So, $G' = H'$ is a countable subgroup of G. For each nonunit element $x \in G'$, there exists a normal subgroup N_x of G with G/N_x a Černikov group and $x \notin N_x$. So, if $N = \bigcap\{N_x : 1 \neq x \in G'\}$, $N \cap G' = 1$. Then, $G \leq (G/G') \times (G/N)$. But $(G/G') \in \mathscr{SD}(C \cup A_0)$ and $\{N_x/N : 1 \neq x \in G'\}$ is a countable residual system for the CC-group G/N. By Theorem 3.1, $G/N \in ZrC$, with a countable number of components and, by Theorem2.5, $G/N \in \mathscr{SD}(C \cup A_0)$, and the proof is complete. □

4. Residually Černikov CC-groups as sections

Gorčakov [5] showed that periodic residually finite FC-groups are sections of direct products of finite groups. Later, this result was a consequence of the complete characterization of the periodic residually finite FC-groups as the subgroups of centrally restricted products of finite groups due to Tomkinson [14]. To date, an analogous characterization has not been obtained for CC-groups. In this section we extend Gorčakov's result, showing that residually Černikov CC-groups are sections of direct products of Černikov and torsion-free abelian groups. In Example 2.4 of [4] there is an example of a CC-group with $G/Z(G)$ non-periodic, and so it is not a section of this type. So, there are CC-groups that are not in the class $\mathscr{QSD}(C \cup A_0)$, and this shows that the hypothesis of residually Černikov cannot be omitted. On the other hand, an infinite countable extra special p-group (see p. 49 of [15]) is $\mathscr{QSD}F$ but it is not residually Černikov. Thus, the classification that we shall obtain in this section is not a characterization because a $\mathscr{QSD}(C \cup A_0)$-group is not always residually Černikov. The next result represents the induction step, and its proof is very close to the corresponding theorem of [5].

Theorem 4.1. *Let G be a CC-group subgroup of the cartesian product*

$$\prod\{F_i : i \in I\}$$

of an uncountable number of Černikov groups F_i. Then G can be embedded as a subgroup of a centrally complete product of CC-groups with cardinal strictly less than $|I|$. Furthermore, if G is periodic, the embedding can be performed in a centrally restricted product.

Now we are able to establish our main result of this section.

Theorem 4.2. *A residually Černikov CC-group is in the class $\mathscr{QSD}(C \cup A_0)$. Furthermore, if G is periodic, G is a $\mathscr{QSD}C$-group.*

Proof. The second statement follows immediately from the first. Let us suppose that $G \leq \prod\{F_i : i \in I\}$, with $|I| \leq |G|$. If I is countable, the result follows

from Corollary 3.2. Let us suppose that $|I|$ is uncountable. By Theorem 4.1, there exists a family $\{G_j : j \in J\}$ of CC-groups with $|G_j| < |I|$ for all j, and such that $G \leq Zr^*G_j$. We can assume that $G_j = G/K_j$ for all $j \in J$, and thus, $G = H_jK_j$ with H_j normal in G and $|H_j| < |I|$. H_j is residually Černikov, and so, by induction $H_j \in \mathcal{QSD}(C \cup A_0)$. Then, Lemma 2.1 implies that $G_j \in \mathcal{QSD}(C \cup A_0)$. Let $Z = \prod Z(G_j)$ and $D = DrG_j$ such that $G \leq Zr^*G_j = ZD$. Then $D \in \mathcal{QSD}(C \cup A_0)$ and so $G \in \mathcal{QSD}(C \cup A_0)$. □

Corollary 4.3. *If G is a CC-group, $G/Z(G) \in \mathcal{QSD}(C \cup A_0)$. If G is periodic, then $G/Z(G) \in \mathcal{QSD}C$.*

Proof. It is a consequence of Theorem 4.2. □

Another traditional step in FC-group theory has been embedding the derived group G' of an FC-group G. Tomkinson [13] has shown that $G' \in \mathcal{QSD}F$, for any FC-group G. Using Theorem 3.1 of [4], we can prove an analogous theorem for CC-groups. The proof is very close to that of Theorem 3.6 of [15].

Theorem 4.4. *If G is a CC-group, then $g' \in \mathcal{QSD}C$.*

We finish this section with some arithmetical properties.

Theorem 4.5. *If G is an infinite residually Černikov CC-group, then $|G| = |G'||Z(G)|$.*

Proof. Proceeding as in Corollary 3.2, $G/N \leq \prod\{G/N_x : 1 \neq x \in G'\}$ with $N = \bigcap\{N_x : 1 \neq x \in G'\}$. If $x \in N$, then $[G, x] \leq N \cap G' = 1$, and so $x \in Z = Z(G)$. Then it is easy to see that $Z(G/N) = Z/N$, and by Lemma 6.3 of [9] $|G/Z| \leq \max\{\aleph_0, |G'|\}$. So, $|G| = |G/Z||Z| \leq \aleph_0|G'||Z|$, and we have equality because G is infinite. Thus, if G' or $Z(G)$ is infinite, the result follows. Otherwise, there exists a normal subgroup K of G with G/K a Černikov group and $K \cap Z \cap G' = 1$. So $K = 1$ and G is Černikov. By Theorem 4.35 of [12], G is an FC-group and so is central by finite. Consequently, G is finite, which is a contradiction, and the proof is complete. □

The result above is false if the group is not residually as shows any infinite extra special p-group.

Corollary 4.6. *If G is a CC-group with $G/Z(G)$ infinite, then $|G| = |G'||Z_2(G)|$.*

Proof. It follows from Theorem 4.5. □

5. Some particular cases of embedding

We show in this section some special cases in which a residually Černikov CC-group can be embedded into a direct product of Černikov and abelian groups. Most of these results are, in the FC-case, consequences of the characterization of residually finite periodic FC-groups given by Tomkinson [14]. First, we present some results which are adaptations of some Gorčakov theorems (see Gorčakov [7] and Theorems 2.2, 2.3, 2.4 of [15]). Since our proofs are very similar to those given there, we just state them. We shall denote by $\pi_J(D_r\{H_i : i \in I\})$ the projection from $D_r\{H_i : i \in I\}$ to $D_r\{H_i : i \in J\}$ for a set of groups $\{H_i : i \in I\}$ and where J is contained in I.

Lemma 5.1. *Let G be such that $G \leq Dr\{H_i : i \in I\}$, where the H_i are groups such that $|H_i| < \kappa$ for all $i \in I$, and where κ is a fixed uncountable cardinal. Then, the index set I can be seen as a union of an ascending chain of subsets $I_{(\alpha)}$, $\alpha < \rho$, ρ being the least ordinal of cardinality $|I|$, such that the following conditions are satisfied.* (a) *For each α, $J_{(\alpha)} = I_{(\alpha+1)} - I_{(\alpha)}$ has cardinality strictly less than κ.* (b) *If $J \subseteq I_{(\alpha)}$ is finite, then $\prod_J(G) = \prod_J(G \cap Dr\{H_i : i \in I_{(\alpha)}\})$.*

Lemma 5.2. *Let G be such that $G \leq Dr\{H_i : i \in I\}$, where the H_i are groups such that $|H_i| < \kappa$ for all $i \in I$ and where κ is an uncountable given cardinal. Then G' is a direct product of normal subgroups of G which have cardinality strictly less than κ.*

As consequences of Lemma 5.1 we can state (see Theorem 2.4 of [15]).

Theorem 5.3. *If $G \in \mathscr{QSD}(C \cup A_0)$, then $G/Z(G) \in \mathscr{SD}C$.*

Corollary 5.4. *If G is a residually Černikov CC-group, then $G/Z(G) \in \mathscr{SD}C$. So, if G is a CC-group, $G/Z_n(G) \in \mathscr{SD}C$ for $n \geq 2$.*

Now we use these previous results to state the more important cases of embedding.

Theorem 5.5. *If G is a residually Černikov CC-group, $G' \in \mathscr{SD}C$.*

Proof. We will proceed by induction on $|G|$. If G is countable, by Corollary 3.2, $G \in \mathscr{SD}(C \cup A_0)$, and since G' is periodic, $G' \in \mathscr{SD}C$. If G is uncountable, by Theorem 4.1 we have $G \leq Zr^*\{G_i : i \in I\}$ with $|I| = |G|$ and G_i is a CC-group with $|G_i| < |G|$. Let $Z = \prod\{Z(G_i) : i \in I\}$ and $D = Dr\{G_i : i \in I\}$. Then $G \leq ZD$. Thus, $G' = (GZ)'$, and since $GZ = (GZ \cap D)Z$, we conclude $G' = (GZ \cap D)'$. By Lemma 5.2 $(GZ \cap D)' = Dr\{H_j : j \in J\}$, where H_j are normal subgroups of $GZ \cap D$ and $|H_j| < |G|$ for all $j \in J$. For each $j \in J$, since $H_j \leq G'$, there exists $K_j \leq G$ such that $H_j \leq K_j'$ and $|K_j| < |G|$. By induction, $K_j' \in \mathscr{SD}C$, and so $H_j \in \mathscr{SD}C$. Thus, $G' \in \mathscr{SD}C$, and the proof is complete. □

Corollary 5.6. *If G is a CC-group, $G'Z(G)/Z(G) \in \mathscr{SD}C$.*

Proof. It is consequence of Theorem 5.5. □

Theorem 5.7. *Let G be a residually Černikov CC-group. If G', G/G' or $Z(G)$ are countable, then $G \in \mathscr{SD}(C \cup A_0)$. Furthermore, if G is periodic, $G \in \mathscr{SD}C$.*

Proof. If G' or $Z(G)$ are countable, proceeding as in Corollary 3.2 and applying Corollary 5.4, we conclude that $G \in \mathscr{SD}(C \cup A_0)$. If G/G' is countable, we prove the result by induction, as in Theorem 5.5. Using the same notation, $G' = Dr\{H_j : j \in J\}$, where H_j are normal subgroups of $GZ \cap D$ and $|H_j| < |G|$ for all $j \in J$. In fact, $H_j \leq G$, H_j are normal subgroups of $(GZ \cap D)Z = GZ$, and so the subgroups H_j are also normal in G. Since G/G' is a countable group, there exists a countable normal subgroup C of G such that $G = CG'$. But $C \cap G'$ is also countable, and so there exists a countable subset $J_0 \subseteq J$ with $C \cap G' = H \leq Dr\{H_j : j \in J_0\}$. Clearly, H is normal in G and, since we are supposing that G is uncountable, $|H| < |G|$. Let $K =: Dr\{H_j : j \in J - J_0\}$ and $E = CH$. Thus, $|E| < |G|$. Then, $E \cap K = CH \cap G' \cap K = (C \cap G')H \cap K = H \cap K = 1$ and, since $G = EK$, we obtain $G = E \times K$. Therefore, $(G/G') = (E/E') \times (K/K')$, and so E/E' and

K/K' are countable. By induction, E and K are $\mathscr{SD}(C \cup A_0)$-groups, and so is G. The second part of the theorem is a consequence of the first, and our result follows. □

In this previous theorem, we showed that a residually Černikov CC-group with $Z(G)$ countable belongs to the class $\mathscr{SD}(C \cup A_0)$. The analogous problem is still unsolved in the FC-case, that is to say, it is unknown if every periodic residually finite FC-group with countable center is in the class $\mathscr{SD}F$. The next result gives a partial answer to this problem.

Theorem 5.8. *Let G be a periodic and residually finite FC-group with countable center $Z(G) = Z$. Then G is isomorphic to a subgroup of a direct product of groups which are all finite except for countably many of them, which are Černikov central by finite groups.*

Proof. We can suppose that $G \leq (G/Z) \times (G/N)$, where G/N is a periodic FC-group with a countable residual system of finite groups. By Corollary 2.26 of [15], $G/Z \in \mathscr{SD}F$. Tomkinson (Theorem 2.24 of [15]) also showed that G/N is a subgroup of a centrally restricted product of a countable number of finite groups. Proceeding as in Theorem 2.5, we can prove that $G/N \leq Dr\{A_i : i \in I\} \times Dr\{C_n : n \geq 1\}$, with A_i Černikov abelian groups and C_n Černikov FC-groups. So, we have $G \leq Dr\{F_j : j \in J\} \times Dr\{A_i : i \in I\} \times Dr\{C_n : n \geq 1\}$ with F_j finite groups. Let $A := Dr\{A_i : i \in I\}$. Clearly, $G \cap A \leq Z$. Since Z is countable, $I_0 = \operatorname{supp}(G \cap A)$ is also countable, but $G \cap (Dr\{A_i : i \in I - I_0\}) = 1$ which implies that $G \leq Dr\{F_j : j \in J\} \times Dr\{A_i : i \in I_0\} \times Dr\{C_n : n \geq 1\}$, and the result follows. □

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