PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 8, August 1995

ON DISGUISED INVERTED WISHART DISTRIBUTION

A. K. GUPTA AND S. OFORI-NYARKO

(Communicated by Wei-Yin Loh)

ABSTRACT. Let $A \sim W_p(n, \Sigma)$ and $A = ZZ'$ where Z is a lower triangular matrix with positive diagonal elements. Further, let $B = A^{-1} = W'W$ have inverted Wishart distribution so that $W = Z^{-1}$. In this paper we derive the distribution of $M = W\Sigma W'$. It is also shown that $\frac{n-p+1}{np} T'MT \sim F_{p, n-p+1}$ where $T \sim N_p(0, I_p)$ is independent of M.

1. INTRODUCTION

While deriving the minimax estimator of a normal covariance matrix when additional information is available on some coordinates [2], we were confronted with finding the distribution of a random variable of the type $(Z'Z)^{-1}$ where Z is a lower triangular matrix with positive diagonal elements such that $ZZ' = A(p \times p) \sim W_p(n, \Sigma)$. Here we first derive the distribution of $M = W\Sigma W'$, where $W = Z^{-1}$. The distribution of $T'MT$ is also derived where independently $T \sim N_p(0, I_p)$. We call M a disguished inverted Wishart variable for reasons explained in §3. Tan and Guttman [3] derived the distribution of a disguised Wishart variable.

In §2 we present some preliminary results which are used in the sequel. In §3 main results of the paper are derived.

2. SOME PRELIMINARY RESULTS

The following lemmas are needed to derive the distribution of M. The proofs of Lemmas 1.1 and 1.2 are given in [1].

Lemma 2.1. *If $M = M_1 M_1'$, where M_1 is a $p \times p$ lower triangular matrix with positive diagonal elements, then*

$$J(M \to M_1) = 2^p \prod_{i=1}^{p} m_{ii(1)}^{p-i+1} \tag{2.1}$$

where $m_{ii(1)}$ is the ith diagonal element of M_1. Also the transformation $N =$

Received by the editors July 7, 1993.
1991 *Mathematics Subject Classification.* Primary 62H10; Secondary 62H12.
Key words and phrases. Minimax estimation, risk, Jacobian, lower triangular matrix, F-distribution.

$M_1'M_1$ *gives*

$$J(N \to M_1) = 2^p \prod_{i=1}^{p} m_{ii(1)}^{i}. \tag{2.2}$$

Lemma 2.2. *If X and A are $p \times p$ lower triangular matrices with positive diagonal elements, then the Jacobian of the transformation $Y = AX$ is*

$$J(Y \to X) = \prod_{i=1}^{p} a_{ii}^{i}, \tag{2.3}$$

and the a_{ii} are the diagonal elements of A.

The following result is available in [3].

Lemma 2.3. *Let $M^{[i]} = (m_{kl})$, $1 \le k, l \le i$, be a submatrix of M. Then*

$$M = M_1 M_1' = \begin{bmatrix} M_{111} & 0 \\ M_{121} & M_{122} \end{bmatrix} \begin{bmatrix} M_{111}' & M_{121}' \\ 0 & M_{122}' \end{bmatrix}$$

gives $M^{[i]} = M_{111}M_{111}'$, with M_{111} as an $i \times i$ principal diagonal block matrix, and $|M^{[i]}| = \prod_{j=1}^{i} m_{jj(1)}^2$. Also $|M^{[i-1]}| = \prod_{j=1}^{i-1} m_{jj(1)}^2$ and thus

$$|M^{[i]}|/|M^{[i-1]}| = m_{ii(1)}^2. \tag{2.4}$$

3. Main results

We prove the main theorem using the three lemmas given in the previous section.

Theorem 3.1. *Let A be distributed as $W_p(n, \Sigma/n)$, where Σ is a positive definite matrix of constants such that $\Sigma = QQ'$, where Q is a lower triangular matrix. Then the distribution of $M = W\Sigma W'$, such that $W'W = A^{-1}$, is given by*

$$f(M) = C_0 \left(\prod_{i=1}^{p} m_{ii(1)}^{2i-p-1} \right) |M|^{-\frac{1}{2}(n+p+1)} e^{-\frac{n}{2}\operatorname{tr} M^{-1}}, \tag{3.1}$$

where $C_0 = n^{np/2}/2^{np/2}\Gamma_p(n/2)$ and $m_{ii(1)}$ is given in (2.4).

Proof. If A is distributed as $W_p(n, \Sigma/n)$, then $B = A^{-1}$ has the density

$$f(B) = \frac{|\Sigma/n|^{-n/2} |B|^{-\frac{1}{2}(n+p+1)} e^{-\frac{1}{2}\operatorname{tr} n\Sigma^{-1}B^{-1}}}{2^{np/2}\Gamma_p(n/2)}. \tag{3.2}$$

Let $B = W'W$, where W is lower triangular matrix. Then

$$f(W) = \frac{|\Sigma/n|^{-n/2} |B|^{-\frac{1}{2}(n+p+1)} \left(2^p \prod_{i=1}^{p} w_{ii}^{i}\right) e^{-\frac{1}{2}\operatorname{tr} n\Sigma^{-1}(W'W)^{-1}}}{2^{np/2}\Gamma_p(n/2)}. \tag{3.3}$$

Since Σ is positive definite, write $\Sigma = QQ'$ where Q is lower triangular matrix. Then, using Lemma 2.1, we get

$$\begin{aligned} f(W) &= C_0 2^p |QQ'|^{-\frac{n}{2}} |W'W|^{-\frac{1}{2}(n+p+1)} \\ &\quad \cdot \left(\prod_{i=1}^{p} w_{ii}^{i} \right) \exp\left\{ -\frac{1}{2} \operatorname{tr} n(QQ')^{-1}(W'W)^{-1} \right\}. \end{aligned} \tag{3.4}$$

Now let $M_1 = WQ$. The Jacobian of this transformation is

$$\left|\frac{\partial W}{\partial M_1}\right| = \prod_{i=1}^{p} q_{ii}^{-(p-i+1)},$$

where q_{ii} is the ith diagonal element of Q. From (3.4) we obtain

$$f(M_1) = C_0 2^p \left(\prod_{i=1}^{p} m_{ii(1)}^{i}\right) |M_1 M_1'|^{-\frac{1}{2}(n+p+1)} e^{-\frac{n}{2}\operatorname{tr}(M_1 M_1')^{-1}},$$

and because both Q and W are lower triangular matrices, M_1 also is a lower triangular matrix. Make the transformation

$$M = M_1 M_1' = (WQ)(WQ)' = W\Sigma W'.$$

From Lemma 2.1, the Jacobian of the transformation $M = M_1 M_1'$ is given by $2^p \prod_{i=1}^{p} m_{ii(1)}^{p-i+1}$, where $m_{ii(1)}$ is the ith diagonal element of M_1 so that the distribution of M is

$$f(M) = C_0 \left(\prod_{i=1}^{p} m_{ii(1)}^{2i-p+1}\right) |M|^{-\frac{1}{2}(n+p+1)} \exp\left(-\frac{n}{2}\operatorname{tr} M^{-1}\right), \tag{3.5}$$

which completes the proof. □

The distribution (3.5) depends on the permutation of M. To illustrate this, let $p = 2$. Then

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} = M_1 M_1' = \begin{pmatrix} m_{11(1)} & 0 \\ m_{12(1)} & m_{22(1)} \end{pmatrix} \begin{pmatrix} m_{11(1)} & m_{12(1)} \\ 0 & m_{22(1)} \end{pmatrix}$$

so that

$$m_{11} = m_{11(1)}^2; \quad m_{12} = m_{11(1)} m_{12(1)}; \quad m_{22} = m_{12(1)}^2 + m_{22(1)}^2. \tag{3.6}$$

Suppose one is interested in a permutation of M, say

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} = \begin{pmatrix} m_{22} & m_{12} \\ m_{12} & m_{11} \end{pmatrix}.$$

Then $C = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) M \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Writing $M = M_1 M_1'$,

$$C = \begin{pmatrix} m_{12(1)}^2 + m_{22(1)}^2 & m_{11(1)} m_{12(1)} \\ m_{11(1)} m_{12(1)} & m_{11(1)}^2 \end{pmatrix}.$$

That is, $c_{11} = m_{12(1)}^2 + m_{22(1)}^2$, $c_{12} = m_{11(1)} m_{12(1)}$, $c_{22} = m_{11(1)}^2$. Recall that

$$c_{11} = m_{22}, \quad c_{12} = m_{12}, \quad c_{22} = m_{11}.$$

We note that these are the same equations as in (3.6), except for appropriate changes in subscripts, and so nothing has really changed.

From (2.4), we have

$$f(m_{11}, m_{12}, m_{22}) = C_0 (m_{11} m_{22} - m_{12}^2)^{-\frac{n+2}{2}} m_{11}^{-1} \exp\left(-\frac{n}{2}(m^{11} + m^{22})\right)$$

where

$$\begin{pmatrix} m^{11} & m^{12} \\ m^{12} & m^{22} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}^{-1}.$$

Similarly,

$$g(c_{11}, c_{12}, c_{22}) = C_0(c_{11}c_{22} - c_{12}^2)^{-\frac{n+2}{2}} c_{22}^{-1} \exp\left(-\frac{n}{2}(c^{11} + c^{22})\right).$$

An interesting observation is that if we make the transformation $N = M_1'M_1$ (recall $M = M_1M_1'$) with $|\partial N/\partial M_1| = 2^p \prod_{i=1}^p m_{ii(1)}^i$,

$$|M_1'M_1| = |M_1M_1'|, \quad \text{and} \quad \operatorname{tr}(M_1'M_1)^{-1} = \operatorname{tr}(M_1M_1')^{-1},$$

then we find from (3.5) that

$$f(N) = C_0|N|^{-\frac{1}{2}(n+p+1)} \exp\left(-\frac{n}{2}\operatorname{tr} N^{-1}\right). \tag{3.7}$$

That is, $N = M_1'M_1 = Q'W'WQ = Q'BQ$ has the inverted Wishart distribution $W^{-1}(n, I_p/n)$. This is the reason why M is called a disguised inverted Wishart variable. Tan and Guttman [3] derived the distribution of a disguised Wishart variable $R=P'VP$ where $PP'=\Sigma^{-1}$ and $\Sigma^{-1} \sim W_p(n-1, V^{-1}/(n-1))$.

Motivated by the work of Tan and Guttman, we now derive the distribution of $G = T'MT$ where $T \sim N_p(0, I_p)$ and is independent of M.

Theorem 3.2. *Let $T \sim N_p(0, I_p)$ such that T and M are independent. Then the distribution of $G = T'MT$ is such that*

$$G \sim \frac{np}{n-p+1} F_{p,n-p+1} \tag{3.8}$$

where $F_{p,n-p+1}$ denotes the F-distribution with $(p, n-p+1)$ degrees of freedom.

Proof. Write

$$G = T'MT = T'M_1M_1'T = T_0'T_0 \tag{3.9}$$

where $T_0 = M_1'T$. Now conditional on M_1,

$$f(T_0|M_1)\alpha|M_1|^{-1} \exp\left(-\frac{1}{2}\operatorname{tr} T_0'(M_1'M_1)^{-1}T_0\right). \tag{3.10}$$

Using (3.4) and (3.10) we have

$$\begin{aligned} f(M_1, T_0)\alpha &\left(\prod_{i=1}^p m_{ii(1)}\right) |M_1|^{-(n+p+2)} \\ &\cdot \exp\left(-\frac{1}{2}\operatorname{tr}[T_0'(M_1'M_1)^{-1}T_0 + n(M_1M_1')^{-1}]\right), \end{aligned}$$

so

$$\begin{aligned} f(T_0)\alpha \int_{M_1} &\left(\prod_{i=1}^p m_{ii(1)}^i\right) |M_1|^{-(n+p+2)} \\ &\cdot \exp\left(-\frac{1}{2}\operatorname{tr}[T_0'(M_1'M_1)^{-1}T_0 + n(M_1M_1')^{-1}]\right) dM_1. \end{aligned} \tag{3.11}$$

Now the exponent in the integrand of (3.11), apart from $(-1/2)$, may be written as

$$\begin{aligned} \operatorname{tr}[T_0'(M_1'M_1)^{-1}T_0 + n(M_1'M_1)^{-1}] &= \operatorname{tr}(M_1')^{-1}[T_0T_0' + nI_p]M_1^{-1} \\ &= \operatorname{tr}(M_1')^{-1}(Q_1'Q_1)M_1^{-1} \end{aligned}$$

where $Q = Q_1'Q_1 = T_0T_0' + nI_p$, Q_1 is lower triangular, and Q is positive definite. To verify this, note that for any $(p \times 1)$ vector $x \neq 0$, $x'Qx = (x'T_0)^2 + nx'x > 0$; hence, Q is positive definite. Therefore, there exists a lower triangular $(p \times p)$ matrix Q_1 with positive diagonal elements such that $Q = Q_1'Q_1$. Thus

$$\begin{aligned}\operatorname{tr}[T_0'(M_1'M_1)^{-1}T_0 + n(M_1'M_1)^{-1}] &= \operatorname{tr}(Q_1M_1^{-1})'(Q_1M_1^{-1}) \\ &= \operatorname{tr}[(M_1Q_1^{-1})]'(M_1Q^{-1})^{-1}.\end{aligned}$$

To evaluate the integral in (3.11), let $U = M_1Q_1^{-1}$. The Jacobian of the transformation is $|\partial U/\partial M_1| = \prod_{i=1}^p q_{ii}^{i-p-1}$. We can rewrite (3.11) as

(3.12)

$$f(T_0)\alpha|Q_1|^{-(n+1)}\int_{U>0}\left(\prod_{i=1}^p u_{ii}^{i-p-1}\right)|U|^{-(n+1)}\exp\left(-\frac{1}{2}\operatorname{tr}(UU')^{-1}\right)dU.$$

The integral in (3.12) is constant. This is easily seen from (3.2) by replacing $n\Sigma^{-1}$ with I_p. Hence

$$f(T_0)\alpha\frac{1}{|Q_1|^{n+1}} = |nI_p + T_0T_0'|^{-((n+1)/2)}.$$

Using the fact that for a $(p \times p)$ matrix M,

$$\int_{R^p}|M + tt'|^{-m/2}\,dt = \frac{\pi^{p/2}\Gamma(\frac{m-p}{2})}{\Gamma(\frac{m}{2})|M|^{(m-1/2)}}$$

and that $\int_{R^p} f(T_0)\,dT_0 = 1$, the density function of T_0 is given by

$$f(T_0) = \frac{\Gamma\left(\frac{n+1}{2}\right)\left(1 + \frac{T_0'T_0}{n}\right)^{(n+1)/2}}{(n\pi)^{p/2}\Gamma\left(\frac{n+1-p}{2}\right)}$$

and therefore $L = ((n+1-p)/2)^{1/2}T_0$ has a multivariate t-distribution with $(n+1-p)$ degrees of freedom. Making the transformation, $x_i = l_i^2/(n-p+1)$, it can be shown that $\prod_{i=1}^p x_i^2 = L'L/(n-p+1)$ is simply the inverted beta distribution $\beta'(p/2, (n+1-p)/2)$ and hence has a $pF_{(p,n-p+1)}/(n-p+1)$ distribution. Therefore

$$\frac{L'L}{n-p+1} \sim \frac{p}{n-p+1}F_{p,n-p+1},$$

i.e.,

$$L'L \sim pF_{p,n-p+1}$$

so that $((n-p+1)/n)T_0'T_0 \sim pF_{p,n-p+1}$ or

$$G = T_0T_0' = T'MT \sim \frac{np}{n-p+1}F_{p,n-p+1}.$$

References

1. W. L. Deemer and I. Olkin, *The Jacobians of certain matrix transformations useful in multivariate analysis*, based on lectures of P. L. Hsu at the University of North Carolina, 1947; Biometrika **38** (1951), 345–367.

2. A. K. Gupta and S. Ofori-Nyarko, *Improved minimax estimators of covariance and precision matrices when additional information is available on some coordinates*, Department of Mathematics and Statistics, Bowling Green State University, Technical Report No. 93–11, 1993.

3. W. Y. Tan and I. Guttman, *A disguised Wishart variable and a related theorem*, J. Roy. Statist. Soc. Ser. B **33** (1971), 147–152.

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403

E-mail address: gupta@andy.bgsu.edu