PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 8, August 1995

# SEMIDIRECT PRODUCTS OF $I-E$ GROUPS

CARTER G. LYONS AND GARY L. PETERSON

(Communicated by Lance W. Small)

ABSTRACT. An $I-E$ group is a group $G$ in which the endomorphism near-ring generated by the inner automorphisms of $G$ equals the endomorphism near-ring generated by the endomorphisms of $G$. In this paper we obtain a result characterizing when a semidirect product of $I-E$ groups of relatively prime orders is an $I-E$ group. We then use this result to show that a semidirect product of cyclic groups of relatively prime orders is an $I-E$ group.

## 1. INTRODUCTION

If $S$ is a semigroup of endomorphisms of an additive (but not necessarily abelian) group $G$, the set of all maps from $G$ to $G$ of the form $\varepsilon_1 s_1 + \cdots + \varepsilon_n s_n$ where $\varepsilon_i = \pm 1$ and $s_i \in S$ forms a distributively generated near-ring under pointwise addition and composition of functions called the *endomorphism near-ring* of $G$ generated by $S$. Indeed, an endomorphism near-ring of $G$ is a subnear-ring of the near-ring $M_0(G)$ consisting of all functions from $G$ to $G$ which fix the identity element of $G$.

When $S = \mathrm{Inn}(G)$ where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of $G$, the endomorphism near-ring generated by $S$ is denoted $I(G)$; when $S = \mathrm{End}(G)$ where $\mathrm{End}(G)$ denotes the semigroup of endomorphisms of $G$, the near-ring generated by $S$ is denoted $E(G)$. We say that $G$ is an $I-E$ group if $I(G) = E(G)$. It is easy to see that an abelian group $G$ is an $I-E$ group if and only if $G$ is cyclic. The first example of a nonabelian $I-E$ group is, in effect, due to Fröhlich in [1] where he showed that if $G$ is a finite nonabelian simple group, then $I(G) = M_0(G)$. Since then other nonabelian $I-E$ groups have come to light. From [2] we see that dihedral groups of order $2n$ are $I-E$ when $n$ is odd. Recently, Malone and Mason have generalized this dihedral result in [4] by showing that a semidirect product of cyclic groups of relatively prime orders is $I-E$ when the cyclic normal subgroup is the commutator subgroup. In [5, Theorem 4.11], it is shown that the symmetric groups $S_n$ are $I-E$ groups when $n \geq 5$.

Noting the groups in [4] and [5] are semidirect products of $I-E$ groups, it is natural to ask whether there are any general conditions under which a

Received by the editors October 12, 1993 and, in revised form, December 6, 1993.
1991 *Mathematics Subject Classification.* Primary 16Y30; Secondary 20E36.
*Key words and phrases.* Endomorphism near-ring, $I-E$ group.

semidirect product of $I-E$ groups is an $I-E$ group. We cannot always have such a semidirect product being an $I-E$ group since dihedral groups of order $2n$ are not $I-E$ when $n$ is even by [3], but what about when the factors have relatively prime orders? In the next section, we shall show that if $G$ is the semidirect product of a normal $I-E$ group $H$ and an $I-E$ group $K$ with $(|H|, |K|) = 1$, the problem of determining when $G$ is $I-E$ reduces to considering whether one endomorphism lies in $I(G)$, namely, the projection of $G$ onto $K$. We then use this result in the third and final section of this paper to prove that the semidirect product of two cyclic groups of relatively prime orders is $I-E$, thereby eliminating the assumption in [4] that the normal cyclic subgroup is the commutator subgroup. However, after we complete this proof, we shall point out that the cyclic subgroups may be reselected so that they still have relatively prime orders and the normal one is the commutator subgroup.

## 2. Semidirect products of $I-E$ groups

As promised in the introduction, we prove the following theorem, the proof of which makes use of Hall $\pi$-subgroups of $\pi$-separable groups. Our reference on these matters will be [6, Section 9.1].

**Theorem 2.1.** *Suppose that $G$ is the semidirect product of a normal subgroup $H$ and a subgroup $K$ where $(|H|, |K|) = 1$ and $H$ and $K$ are $I-E$ groups. If $\varepsilon$ is the projection from $G$ onto $K$ given by $(h+k)\varepsilon = k$ where $h \in H$ and $k \in K$, then $G$ is an $I-E$ group if and only if $\varepsilon \in I(G)$.*

*Proof.* Of course the endomorphism $\varepsilon$ is certainly in $I(G)$ when $G$ is an $I-E$ group, so that the main part of the proof is to obtain the converse. To this end, let $\pi$ denote the set of primes of $|H|$ in which case $H$ is a Hall $\pi$-subgroup of $G$, $K$ is a Hall $\pi'$-subgroup of $G$, and $G$ is $\pi$-separable.

Let $\alpha \in \mathrm{End}(G)$. We must show $\alpha \in I(G)$. Write

$$\alpha = (1-\varepsilon)\alpha + \varepsilon\alpha.$$

Note that $H$ is the unique Hall $\pi$-subgroup of $G$ since all Hall $\pi$-subgroups of $G$ are conjugate [6, 9.1.6]. Since $H\alpha$ is a $\pi$-subgroup of $G$, another application of 9.1.6 of [6] gives us that $H\alpha$ lies in some Hall $\pi$-subgroup of $G$ and hence $H\alpha \subseteq H$. Thus

$$\alpha\,|_H = (1-\varepsilon)\alpha\,|_H \in \mathrm{End}(H).$$

Since $H$ is an $I-E$ group, we have

$$(1-\varepsilon)\alpha\,|_H = \hat{r}$$

for some $\hat{r} \in I(H)$. As $\hat{r}$ consists of sums and differences of inner automorphisms of $H$, we may view $\hat{r} = r\,|_H$ where $r \in I(G)$. Now we have

$$(1-\varepsilon)\alpha = (1-\varepsilon)r,$$

for if $g \in G$, writing $g = h + k$ where $h \in H$ and $k \in K$ gives us

$$g(1-\varepsilon)\alpha = h\alpha = h\hat{r} = g(1-\varepsilon)r.$$

Hence $(1-\varepsilon)\alpha = (1-\varepsilon)r \in I(G)$.

Next we consider $\varepsilon\alpha$. Note that $G\varepsilon\alpha$ is a $\pi'$-subgroup of $G$. Applying 9.1.6 of [6] yet one more time, we have that $G\varepsilon\alpha$ is contained is some Hall $\pi'$-subgroup $K'$ of $G$ and that $K = (K')^g$ for some $g \in G$. Letting $\tau$ denote the inner automorphism of $G$ induced by $g$ and $\beta = \alpha\tau$, the proof will be complete if we show $\varepsilon\beta \in I(G)$.

We have $\varepsilon\beta\,|_K \in \mathrm{End}(K)$ and hence

$$\varepsilon\beta\,|_K = \hat{s}$$

where $\hat{s} \in I(K)$. As we did with $\hat{r}$ earlier, we have $\hat{s} = s\,|_K$ for some $s \in I(G)$ and we find $\varepsilon\beta = \varepsilon s \in I(G)$, completing the proof. □

A corollary we shall need in the next section is:

**Corollary 2.2.** *If $G$ is the direct product of $H$ and $K$ where both $H$ and $K$ are $I-E$ groups and if $(|H|, |K|) = 1$, then $G$ is an $I-E$ group.*

*Proof.* Suppose $|H| = n$ and $|K| = m$. Letting $k$ be an integer such that $kn \equiv 1 \bmod m$, we have $\varepsilon = kn \cdot 1 \in I(G)$. □

## 3. Semidirect products of cyclic groups

Suppose that our group $G$ is a semidirect product of a cyclic normal subgroup $H$ and a cyclic subgroup $K$. Our goal in this section is to show that if $(|H|, |K|) = 1$, then $G$ is an $I-E$ group. Since $H$ and $K$ are $I-E$ groups, we will want to show that the projection $\varepsilon$ of Theorem 2.1 is in $I(G)$.

We will need a preliminary lemma in the case when $H$ is a $p$-group. Its proof will depend on some facts about the automorphism group of $H$, $\mathrm{Aut}(H)$, when $H$ is a cyclic $p$-group which can be gleaned from pages 120 and 121 of [7]. First, if $|H| = p^n$, then

$$|\mathrm{Aut}(H)| = p^{n-1}(p-1).$$

In addition, if $p$ is odd and $H = \langle a \rangle$:

(i) $\mathrm{Aut}(H) = \langle \alpha \rangle \times \langle \beta \rangle$ with $a\alpha = (1+p)a$ and $a\beta = ka$ where $k \in \mathbb{Z}$ is a unit of order $p-1$ in $\mathbb{Z}_{p^n}$.

(ii) $|\alpha| = p^{n-1}$ and $|\beta| = p-1$.

Further, an examination of the construction of $\beta$ reveals:

(iii) $k$ is a unit of order $p-1$ in $\mathbb{Z}_p$.

To see (iii), we observe that $\beta$ is constructed by first choosing an integer $m$ which generates the unit group of $\mathbb{Z}_p$ so that $m$ has order $p-1$ modulo $p$. In the unit group of $\mathbb{Z}_{p^n}$ which has order $p^{n-1}(p-1)$, $m$ has order $p^i(p-1)$ for some $0 \le i < n$. We then use $m^{p^i}$ for $k$. The order of $k$ in the unit group of $\mathbb{Z}_p$ is $p-1$ by Fermat's Little Theorem.

**Lemma 3.1.** *Suppose that $G$ is the semidirect product of a normal subgroup $H$ and a subgroup $K$ where $H$ is a cyclic $p$-group and $(|H|, |K|) = 1$. If $b \in K$, then either $[H, b] = 0$ or $[H, b] = H$ where $[H, b]$ denotes the set of commutators $[h, b]$, $h \in H$.*

*Proof.* Suppose $[H, b] \neq 0$. Let $H = \langle a \rangle$ and $|H| = p^n$. If $\tau$ denotes the automorphism that $b$ induces on $H$ by conjugation, then $|\tau|$ divides $p-1$ since $(|H|, |K|) = 1$, and hence $p$ is odd. Also, using the notation preceeding this lemma, $\tau \in \langle \beta \rangle$. Suppose $a\tau = ja$ where $j$ is an integer. Then $j = k^l$

where $0 < l < p-1$ by (i). By (iii), we have $j = s + tp$ with $s, t \in \mathbb{Z}$ where $1 < s < p$. Hence $j - 1 = s - 1 + tp$ is relatively prime to $p$, and consequently

$$[a, b] = -a + a\tau = (j-1)a$$

is a generator for $H$. From the commutator identity

$$(1) \qquad [x + y, z] = [x, z]^y + [y, z],$$

it follows that $[H, b] = H$. □

Now we are ready for:

**Theorem 3.2.** *If $G$ is the semidirect product of a cyclic normal subgroup $H$ and a cyclic subgroup $K$ where $(|H|, |K|) = 1$, then $G$ is an $I$-$E$ group.*

*Proof.* Suppose $H = \langle a \rangle$ and $K = \langle b \rangle$. We have that if $P$ is a Sylow subgroup of $H$, then either $[P, b] = 0$ or $[P, b] = P$ by Lemma 3.1. Let $H_1$ denote the sum of such Sylow subgroups $P$ with $[P, b] = 0$ and $H_2$ the sum of those with $[P, b] = P$. Note that $G$ is the direct product of $H_2 + K$ and $H_1$. Further $(|H_2 + K|, |H_1|) = 1$, so that by Corollary 2.2 it suffices to show $H_2 + K$ is an $I$-$E$ group. To put it another way, we might as well assume $H_1 = 0$ and $H_2 = H$.

Since $[P, b] = P$ for each Sylow subgroup $P$ of $H$, it follows from the commutator identity in equation (1) that $[H, b] = H$ and that $[a, b]$ generates $H$. Writing $a^b = ja$ where $j$ is an integer in which case $[a, b] = (j-1)a$ and denoting the order of $H$ by $n$, we then have that $(j-1, n) = 1$. Letting $m$ be an integer such that $m(j-1) \equiv 1 \bmod n$, $\tau$ denote the inner automorphism that $b$ induces on $G$, and $\varepsilon$ be the projection in Theorem 2.1, the reader can check that $\varepsilon = -jm(\tau - 1) + \tau$. Thus $\varepsilon \in I(G)$ as required. □

Looking back through the proof of Theorem 3.2, note that $G$ is the semidirect product of the cyclic subgroups $H_2$ and $K + H_1$ which have relatively prime orders and that $G' = H_2$. Consequently, Malone and Mason did have Theorem 3.2 in [4], but were unaware of it.

## References


1. A. Fröhlich, *The near-ring generated by the inner automorphisms of a finite simple group*, J. London Math. Soc. **33** (1958), 95–107.
2. C. G. Lyons and J. J. Malone, *Finite dihedral groups and d. g. near-rings.* I, Compositio Math. **24** (1972), 305–312.
3. ———, *Finite dihedral groups and d. g. near-rings.* II, Compositio Math. **26** (1973), 249–259.
4. J. J. Malone and G. Mason, *$ZS$-metacyclic groups and their endomorphism near-rings*, Monatsh. Math. (to appear).
5. J. D. P. Meldrum, *On the structure of morphism near-rings*, Proc. Roy. Soc. Edinburgh Sect. A **81** (1978), 287–298.
6. D. J. S. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1982.
7. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, NJ, 1964.



Department of Mathematics, James Madison University, Harrisonburg, Virginia 22807

*E-mail address*: lyons@math.jmu.edu

*E-mail address*: peterson@math.jmu.edu