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NON-EXTENDIBILITY OF THE BERS ISOMORPHISM

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ABSTRACT. Let G be a torsion free finitely generated Fuchsian group of the first kind of type (p, n). The purpose of this paper is to show that the Bers isomorphism of the Bers fiber space $F(G)$ onto the Bers embedding of $T(\dot{G})$ has no continuous extension to the boundary, provided that $\dim T(G) \geq 1$, where $\dot{G}$ is another torsion free finitely generated Fuchsian group of the first kind of type $(p, n+1)$.

1. INTRODUCTION AND PRELIMINARIES

Let G be a torsion free finitely generated Fuchsian group of the first kind acting on the upper half plane U. Assume that U/G is of type (p, n), where p is the genus of U/G and n is the number of the punctures on U/G.

Let $M(G)$ denote the space of measurable functions μ on U satisfying the conditions

(1) $\|\mu\|_\infty < 1$ and

(2) $(\mu \circ g) \cdot \overline{g}'/g' = \mu$, for all $g \in G$.

Two elements $\mu, \mu' \in M(G)$ are equivalent if $w^\mu = w^{\mu'}$ on $\widehat{\mathbb{R}}$, where w^μ is the unique quasiconformal mapping on $\widehat{\mathbb{C}}$ which fixes $0, 1, \infty$, is conformal on the lower half plane L and satisfies the Beltrami equation $w_{\overline{z}} = \mu w_z$ on U. The equivalence class of μ is denoted by $[\mu]$. The *Teichmüller space* $T(G)$ of G is the space of equivalence classes $[\mu]$ for $\mu \in M(G)$.

The *Bers fiber space* $F(G)$ over $T(G)$ is, by definition of Bers [2], a subset of $T(G) \times \widehat{\mathbb{C}}$ consisting of pairs $([\mu], z)$, where $[\mu] \in T(G)$ and $z \in w^\mu(U)$.

Choose an arbitrary $a \in U$, and let $A = G(a) = \{g(a);\, g \in G\}$. Let

$$h\colon U \to U - A$$

be a holomorphic universal covering. The Fuchsian model for the action of G on $U - A$ is the group

$$\dot{G} = \{\dot{g} \in \operatorname{Aut} U;\ \text{there is a } g \in G \text{ with } h \circ \dot{g} = g \circ h\}.$$

It is easy to see that $U/\dot{G} = U/G - \pi(a)$, where $\pi\colon U \to U/G$ is, as usual, the natural projection.

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Every point in $F(G)$ is represented as a pair $([\mu], w^\mu(a))$ for some $\mu \in M(G)$ by Lemma 6.3 of Bers [2]. On the other hand, we can define a surjective map $h^*: M(\dot{G}) \to M(G)$ by the formula

$$(h^*(\nu)) \circ h = \nu \cdot (h'/\overline{h}'), \quad \text{for } \nu \in M(\dot{G}).$$

Fix $x \in F(G)$, and write $x = ([\mu], w^\mu(a))$. Since h^* is surjective, there is a $\nu \in M(\dot{G})$ such that $h^*(\nu) = \mu$. We define $\varphi: F(G) \to T(\dot{G})$ by sending x to $[\nu]$. φ is well defined by Lemma 6.10 of Bers [2]. Furthermore, the important Bers isomorphism theorem (Theorem 9 of Bers [2]) asserts that φ is a biholomorphic map.

Let $B_2(L, G)$ denote the Banach space consisting of all holomorphic functions ϕ defined on L such that

$$(\phi \circ g)(z)(g')^2(z) = \phi(z), \quad \text{for all } g \in G \text{ and } z \in L,$$

and

$$\sup\{|z - \overline{z}|^2 |\phi(z)|;\ z \in L\} < \infty.$$

The *Bers embedding* of $T(G)$ into $B_2(L, G)$ is given by

$$T(G) \ni [\mu] \mapsto S_{w^\mu|L} \in B_2(L, G),$$

where S_f is the Schwarzian derivative of f. In what follows, we identify $T(G)$ with its image of the Bers embedding. $T(G)$ is a bounded domain in $B_2(L, G)$. For more details, see Bers [2].

Similarly, we may define the Bers embedding of $T(\dot{G})$ into $B_2(L, \dot{G})$. Since $F(G)$ is a domain of $B_2(L, G)\times\widehat{\mathbb{C}}$ and $T(\dot{G})$ is a bounded domain in $B_2(L, \dot{G})$, the topological boundaries of $F(G)$ and $T(G)$ are naturally defined. Let $\overline{F(G)}$ denote the closure of $F(G)$. I. Kra has asked if the Bers isomorphism of $F(G)$ onto $T(\dot{G}) \subset B_2(L, \dot{G})$ has a continuous extension to $\overline{F(G)}$. The purpose of this paper is to settle this problem in the negative if U/G is not of type $(0, 3)$. The main result is the following.

Theorem 1. *Suppose that* $\dim T(G) \geq 1$. *Then the Bers isomorphism of* $F(G)$ *onto* $T(\dot{G})$ *cannot be continuously extended to the closure of* $F(G)$.

Theorem 1 can be generalized as

Theorem 2. *With the conditions of Theorem* 1, *every biholomorphic map of* $F(G)$ *onto* $T(\dot{G})$ *admits no continuous extensions to* $\overline{F(G)}$.

Remark. If $\dim T(G) = 0$, that is, U/G is of type $(0, 3)$, then $F(G)$ is a disc D. It is well known that $T(\dot{G}) = D'$ is a simply connected domain in $B_2(L, \dot{G})$ (the dimension of $B_2(L, \dot{G})$ is one). Any conformal mapping f of D onto D' can be continuously extended if and only if D' has a locally connected boundary. An interesting question as to whether f has a continuous extension remains open.

2. Two lemmas

Let G be a torsion free finitely generated Fuchsian group of the first kind. A hyperbolic element $g \in G$ is called *essential* if the projection C of the axis of g under the natural projection $\pi: U \to U/G$ is a filling curve; that is, $U/G - C$ is the union of discs and punctured discs. (For an equivalent definition, see Kra [6].) The author thanks the referee for providing a short proof of Lemma 1, which appears here.

Lemma 1. *Let G be a torsion free finitely generated Fuchsian group of the first kind. The set of fixed points of essential hyperbolic elements of G are dense in $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.*

Proof. It is well known that for any point $x \in \widehat{\mathbb{R}}$, the orbit $G(x) = \{g(x);\, g \in G\}$ is dense. In particular, the orbit of the fixed point of an essential hyperbolic element g (always exists) is dense; these are fixed points of conjugates of g, which are again essential hyperbolic elements.

Let $Q(G)$ be the group of quasiconformal self-homeomorphisms w of U such that $w \circ g \circ w^{-1} \in PSL(2, \mathbb{R})$ for any $g \in G$. Let $N(G)$ denote the normalizer of G in $Q(G)$. An automorphism θ of G is called *geometric* if there is an element $w \in N(G)$ such that $\theta(g) = w \circ g \circ w^{-1}$ for all $g \in G$. The modular group $\operatorname{mod} G$ of G is defined as a group of geometric automorphisms. Every element $\theta \in \operatorname{mod} G$ acts on $F(G)$ in the following way: Let $w \in N(G)$ be such that $\theta(g) = w \circ g \circ w^{-1}$ for all $g \in G$. Then

$$\theta([\mu], z) = ([\nu], \hat{z}),$$

where ν is the Beltrami coefficient of the map $w^\mu \circ w^{-1}$ and $\hat{z} = w^\nu \circ w \circ (w^\mu)^{-1}(z)$. $\operatorname{mod} G$ is a group of fiber-preserving holomorphic automorphisms of $F(G)$. By Theorem 10 of Bers [2], $\operatorname{mod} G$ is isomorphic to a subgroup of the Teichmüller modular group $\operatorname{Mod} \dot{G} = \operatorname{mod} \dot{G}/\dot{G}$ with finite index $n+1$, where n is the number of the punctures on U/G. More precisely, the elements of the image of $\operatorname{mod} G$ are induced by those quasiconformal homeomorphisms which fix one special puncture of $U/\dot{G}$. It is easy to check that G is a normal subgroup of $\operatorname{mod} G$. We regard G as a subgroup of the Teichmüller modular group $\operatorname{Mod} \dot{G}$ in this natural manner.

Lemma 2. *Suppose that $\dim T(G) \geq 1$. There is no continuous injective map $\tilde{\varphi}$ of $\overline{F(G)}$ into $T(\dot{G}) \cup \partial T(\dot{G})$ extending φ.*

Proof. Let $g \in G$ be an essential hyperbolic element. Under the inclusion $G \subset \operatorname{mod} G \hookrightarrow \operatorname{Mod} \dot{G}$ described above, $\varphi \circ g \circ \varphi^{-1} = \chi_g$ is an element of $\operatorname{Mod} \dot{G}$. By Theorem 2 of Kra [6], χ_g is a hyperbolic modular transformation (in the sense of Bers [4]).

Choose two points $([\mu_1], z_1)$, $([\mu_2], z_2)$ in $F(G)$ lying in different fibers, that is, $w^{\mu_1} \neq w^{\mu_2}$ on $\widehat{\mathbb{R}}$. Let us consider the two sequences $\{g^n([\mu_i], z_i)\}$, for $i = 1, 2$. Observe that the g^n-action on $F(G)$ is defined in a quite natural way; that is,

$$\begin{aligned} g^n([\mu_i], z_i) &= ([\mu_i], (g^{\mu_i})^n(z_i)) \\ &= ([\mu_i], w^{\mu_i} \circ g^n \circ (w^{\mu_i})^{-1}(z_i)), \end{aligned}$$

for $i = 1, 2$. It follows that the action of g^n keeps both fibers $([\mu_1], w^{\mu_1}(U))$ and $([\mu_2], w^{\mu_2}(U))$ invariant. Observe also that on the fiber over $[\mu]$, g^n acts as a hyperbolic Möbius transformation g^μ in the quasi-Fuchsian group $G^\mu = w^\mu G (w^\mu)^{-1}$. Therefore, the sequence $\{(g^{\mu_1})^n(z_1)\}$ must converge to the attractive fixed point of g^{μ_1}, say z_1', lying in the quasicircle $w^{\mu_1}(\widehat{\mathbb{R}})$. Similarly, the sequence $\{(g^{\mu_2})^n(z_2)\}$ converges to the attractive fixed point z_2' of g^{μ_2} lying in $w^{\mu_2}(\widehat{\mathbb{R}})$. Since $\{g^n([\mu_1], z_1)\}$ and $\{g^n[(\mu_2], z_2')\}$ lie in two different fibers, these two sequences converge to two different limit points $([\mu_1], z_1')$ and $([\mu_2], z_2')$. (Note that z_1' and z_2' may coincide.)

For $i = 1, 2$, let $[\nu_i]$ denote the φ-image of $([\mu_i], z_i)$ in $T(\dot{G})$. We consider the sequence $\{\chi_g^n([\nu_i])\}$. For any $n \geq 1$, we have

$$\chi_g^n([\nu_i]) = \varphi \circ g^n \circ \varphi^{-1}([\nu_i]) = \varphi(g^n([\mu_i], z_i)).$$

This implies that $\{\chi_g^n([\nu_i])\}$ is the φ-image of $\{g^n([\mu_i], z_i)\}$. Selecting if need be a subsequence, we may assume that both sequences, $\{\chi_g^n([\nu_i])\} = \varphi(g^n([\mu_i], z_i))$, $i = 1, 2$, are convergent. By the theorem of Bers [3], both sequences converge to the same boundary point which represents a totally degenerate b-group G' isomorphic to G. If φ can be extended injectively to $\overline{F(G)}$, then

$$\tilde{\varphi}([\mu_i], z_1') \neq \tilde{\varphi}([\mu_2], z_2').$$

This is a contradiction. Hence, the lemma is proved.

Remark. By using Lemma 2 we can easily solve the problem related to the inverse of φ; namely, we claim that there is no continuous extension of φ^{-1} to the closure $T(\dot{G}) \cup \partial T(\dot{G})$. Indeed, as we saw before, for $i = 1, 2$, the sequence $\{\chi_g^n([\nu_i])\}$ is the φ-image of $\{g^n([\mu_i], z_i)\}$. Suppose that $\tilde{\varphi}^{-1}$ is continuous; since (choose a subsequence if necessary)

$$\lim_{n\to\infty} \chi_g^n([\nu_1]) = \lim_{n\to\infty} \chi_g^n([\nu_2]) = \phi',$$

where ϕ' corresponds to a totally degenerate b-group $G' = W_{\phi'} G W_{\phi'}^{-1}$. We must have

$$\lim_{n\to\infty} g^n([\mu_1], z_1) = \lim_{n\to\infty} g^n([\mu_2], z_2).$$

Thus,

$$([\mu_1], z_1') = ([\mu_2], z_2'),$$

but this is a contradiction; proving our assertion. To obtain the same conclusion for the isomorphism φ, we must do some further work.

3. Proof of Theorem 1

First, we prove the theorem under the assumption that $\dim T(G) \geq 2$. Suppose that there is a continuous extension $\tilde{\varphi}$ of φ to the closure $\overline{F(G)}$. Let $\alpha \in G$ be any simple hyperbolic Möbius transformation; that is, the projection of the axis $A(\alpha)$ of α under π is a simple closed geodesic on $S = U/G$. By Theorem 2 of Kra [6], as an element of $\operatorname{Mod} \dot{G}$, $\varphi \circ \alpha \circ \varphi^{-1} = \chi_\alpha \in \operatorname{Mod} \dot{G}$ is a parabolic modular transformation in the sense of Bers [4].

To proceed, we need to investigate more carefully the action of the parabolic modular transformations χ_α which are determined by simple hyperbolic transformations α of G. We invoke Theorem 2 of Nag [8], which says that the self-homeomorphism f_α which induces χ_α is isotopic to a spin about $\hat{a}$, where $\hat{a}$ is the projection of a (defined in Section 1) under $\pi: U \to U/G$. This means that the system of admissible curves defined by f_α is $C = \{C_1, C_2\}$, where C_1 and C_2 bound a cylinder A containing the puncture $\hat{a}$ and no other punctures. Further, since α is hyperbolic, neither C_1 nor C_2 bounds a punctured disk.

On the other hand, we know that the number of the curves in a maximal system for $U/\dot{G}$ is $3p - 2 + n$ (where (p, n) is the type of G), and that f_α is reduced by a system with two simple closed curves $C = \{C_1, C_2\}$. Thus, C

is not of maximal system unless $\dim T(G) = 0$ or 1; that is, unless $(p, n) = (0, 3), (0, 4)$, or $(1, 1)$.

For any $x \in T(\dot{G})$, let us consider the set $A(\chi_\alpha, x)$ of accumulation points of $\{\chi_\alpha^n(x)\}$. By Theorem 3 of Abikoff [1], $A(\chi_\alpha, x)$ consists of those quadratic differentials ϕ in $B_2(L, \dot{G})$ for which $W_\phi \dot{G} W_\phi^{-1}$ are nondegenerate cusps (that is, regular b-groups).

Fix $x \in T(\dot{G})$; by passing to a subsequence if necessary, we assume that $\{\chi_\alpha^n(x)\}$ converges. This implies that $A(\chi_\alpha, x)$ consists of only one point, which corresponds to a regular b-group, say B. Topologically, the upper structure $(\Omega(B)-\Delta(B))/B$ of B (where $\Omega(B)$ is the discontinuous region of B and $\Delta(B)$ is the simply connected invariant domain of B) is obtained by squeezing the curves C on $U/\dot{G}$. See Theorem 5 of Maskit [7].

By the previous argument, we see that $U/\dot{G} - C$ consists of two or three components S_i. (The number depends on whether $U/\dot{G} - A$ is connected or disconnected.) We also know that at least one component is not a pair of pants. Therefore, we can change the conformal structure on $\widehat{S}_1 + \cdots + \widehat{S}_m$, $m = 2$ or 3, where $\widehat{S}_i$ are obtained from S_i by capping the punctured discs on the boundary curves. Fix a conformal structure on $\widehat{S}_1 + \cdots + \widehat{S}_m$, and use the same notation; from Theorem 6 of Maskit [7], we conclude that there is a regular b-group B lying on the boundary of $T(\dot{G})$ such that $(\Omega(B) - \Delta(B))/B = \widehat{S}_1 + \cdots + \widehat{S}_m$. Different conformal structures on $\widehat{S}_1 + \cdots + \widehat{S}_m$ will produce different regular b-groups. Let B, B_0 be two distinct regular b-groups defined in this way, and let $\phi, \phi_0 \in B_2(L, \dot{G})$ be the quadratic differentials corresponding to B and B_0, respectively; that is, $B = W_\phi \dot{G} W_\phi^{-1}$ and $B_0 = W_{\phi_0} \dot{G} W_{\phi_0}^{-1}$.

By selecting a further subsequence, we assume that the sequence $\{\chi_\alpha^n\}$ of bounded analytic maps converges. Theorem 3 of Abikoff [1] then asserts that $\{\chi_\alpha^n\}$ converges to a limiting holomorphic map of $T(\dot{G})$ to $\partial T(\dot{G})$ which is a surjection of $T(\dot{G})$ onto the boundary Teichmüller space representing the corresponding congruence class (for the definition, see Abikoff [1]). This implies that there are points $x, y \in T(\dot{G})$ such that $\{\chi_\alpha^n(x)\}$ converges to ϕ and $\{\chi_\alpha^n(y)\}$ converges to ϕ_0.

Let $([\mu_1], z_1)$ and $([\mu_2], z_2) \in F(G)$ denote the preimages of x and y under $\varphi: F(G) \to T(\dot{G})$, respectively. There are two cases.

Case 1. μ_1 is not equivalent to μ_2; that is, $([\mu_i], z_i)$, $i = 1, 2$, lie in different fibers. Consider the sequence $\{\alpha^n([\mu_i], z_i)\}$; these sequences are the preimages of $\{(\chi_\alpha)^n(x)\}$ and $\{(\chi_\alpha)^n(y)\}$. By using the same proof as in Lemma 2, we conclude that the limit points z_i' of $\{\alpha^n([\mu_i], z_i)\}$, $i = 1, 2$, lie in the boundaries of different fibers, $([\mu_i], w^{\mu_i}(\widehat{\mathbb{R}}))$, and the images of z_1' and z_2' under $\tilde{\varphi}$ (we assume that there is a continuous extension $\tilde{\varphi}$ of φ) is exactly ϕ and ϕ_0 described above. By Lemma 1, we can choose a sequence $\{u_n\}$ of fixed points of essential hyperbolic Möbius transformations in G such that $\{u_n\}$ converges to a fixed point z' of $\alpha \in G$. But $w^{\mu_i}(z')$ is a fixed point of $\alpha^{\mu_i} \in G^{\mu_i}$, which is equal to z_i'. It follows that $(w^{\mu_1})^{-1}(z_1') = (w^{\mu_2})^{-1}(z_2')$. Let $\{\theta_n\}$, $n = 1, 2, \ldots$, denote the corresponding essential hyperbolic elements of G. Since w^{μ_i}, $i = 1, 2$, are global homeomorphisms, the sequences $\{u_{i,n}\}$ of the fixed points of $\{\theta_n^{\mu_i}\}$ also converge to z_i'. For $i = 1, 2$, choose $y_i \in F(G)$ so that y_i lie in the fibers $([\mu_i], w^{\mu_i}(U))$, respectively. Since $\{\theta_n\} \subset G \hookrightarrow$

mod G, if we fix n, then the sequence $\{\theta_n^m(y_i)\}$ converges to $u_{i,n}$, since $u_{i,n}$ is a fixed point of $\theta_n^{\mu_i}$ (if $u_{i,n}$ is the repulsive fixed point, then we replace m by $-m$, and the above argument still works). We denote by x_i the φ-image of y_i in $T(\dot{G})$ for $i = 1, 2$. The sequences $\{\theta_n^m(y_i)\}$ are mapped via φ to the sequences $\{\chi_{\theta_n}^m(x_i)\}$. By selecting a subsequence if necessary, we may assume that the two sequences $\{\chi_{\theta_n}^m(x_i)\}$, $i = 1, 2$, converge for every $n \in \mathbb{Z}^+$. By using the same proof as in Lemma 2, we conclude that for $i = 1, 2$ and a fixed n, the two sequences $\{\chi_{\theta_n}^m(x_i)\}$ converge to a single point ϕ_n. Let $G_n = W_{\phi_n}\dot{G}W_{\phi_n}^{-1}$. Then all G_n are, by the theorem of Bers [3], totally degenerate b-groups in $\partial T(\dot{G})$ isomorphic to $\dot{G}$. It follows that if the continuous extension $\tilde{\varphi}$ of φ is possible, then we must have

$$\tilde{\varphi}(u_{1,n}) = \tilde{\varphi}(u_{2,n}) = \phi_n.$$

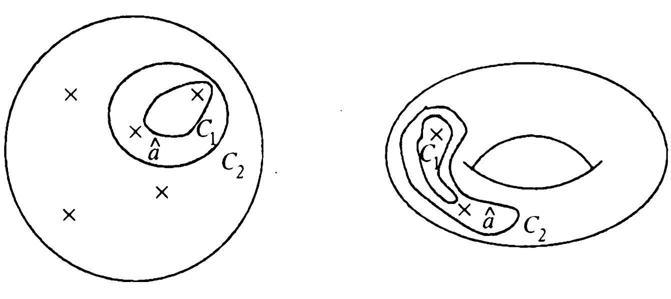
Since $\{u_{i,n}\}$, $i = 1, 2$, converge to z_i', and since $\tilde{\varphi}(z_1') = \phi$ and $\tilde{\varphi}(z_2') = \phi_0$, $\{\phi_n\}$ must converge to both ϕ and ϕ_0. This is clearly impossible.

Case 2. μ_1 is equivalent to μ_2. In this case y_i, $i = 1, 2$, lie in the same fiber. This means that the sequence $\{\theta_n^m(y_i)\}$ converges to $u_n \in \partial w^{\mu_1}(\widehat{\mathbb{R}})$ (n is fixed). It follows that $\tilde{\varphi}(u_n)$ is the limit ϕ_n of the sequence $\{\chi_{\theta_n}^m(\varphi(y_i))\}$. Since $\{u_n\}$ converges to $z_1' = z_2'$, $\tilde{\varphi}(u_n) = \phi_n$ converges to ϕ. Similarly, $\tilde{\varphi}(u_n) = \phi_n$ also converges to ϕ_0. This is impossible.

Next, we deal with the case of $\dim T(G) = 1$; that is, U/G is of type $(0, 4)$ or $(1, 1)$. This means that the type of $U/\dot{G}$ is $(0, 5)$ or $(1, 2)$. Choose a spin $s = h_{C_2} \circ h_{C_1}^{-1}$ about the puncture $\hat{a}$ (recall that $\hat{a}$ is the projection of a under $\pi: U \to U/G$), where h_{C_i} is the Dehn twist about a simple closed curve C_i, and C_1 bounds a punctured disk (see Figure 1). In this case, the spin s defined on $U/\dot{G}$ is isotopic to the Dehn twist about C_2. (The Dehn twist about C_1 is isotopic to the identity.) Let $\chi \in \operatorname{Mod}\dot{G}$ be the (parabolic) modular transformation induced by s. By selecting a subsequence if necessary, we see that $\{\chi^n(x)\}$, $x \in T(\dot{G})$, converges to a quadratic differential ϕ' corresponding to a regular b-group B' which satisfies

$$(\Omega(B') - \Delta(B'))/B' = \widehat{S}_1' + \widehat{S}_2',$$

where $\widehat{S}_1'$ is a thrice punctured sphere, $\widehat{S}_2'$ is a 4-times punctured sphere if $U/\dot{G}$ is of type $(0, 5)$ and is a punctured torus if $U/\dot{G}$ is of type $(1, 2)$. In



(a) (b)

FIGURE 1

both cases, $\widehat{S}'_2$ has moduli. Thus, we can change the conformal structure on $\widehat{S}'_1 + \widehat{S}'_2$.

On the other hand, since χ is induced by s and s fixes $\hat{a}$, by Theorem 10 of Bers [2], $\varphi^{-1} \circ \chi \circ \varphi \in \operatorname{mod} G$. Note that the following diagram is commutative:

$$\begin{array}{ccc} T(\dot{G}) & \xrightarrow{\chi} & T(\dot{G}) \\ \pi_0 \downarrow & & \pi_0 \downarrow \\ T(G) & \xrightarrow{\text{id}} & T(G) \end{array}$$

where $\pi_0 = \pi_G \circ \varphi^{-1}$ and $\pi_G \colon F(G) \to T(G)$ is the natural projection. We conclude that $\varphi^{-1} \circ \chi \circ \varphi = \alpha \in G$. It is easy to see that α is a parabolic element of G. Instead of choosing a simple hyperbolic element of G, we choose α as our original element; the argument of this section works equally well in this case. The details are omitted.

4. Proof of Theorem 2 (sketch)

Suppose that $\psi \colon F(G) \to T(\dot{G})$ is a biholomorphic map which can be extended continuously to the boundary. Then $\psi \circ \varphi^{-1}$ is a holomorphic automorphism of $T(\dot{G})$. From a theorem of Royden [9] (its generalization is due to Earle-Kra [5]), $\psi \circ \varphi^{-1} \in \operatorname{Mod} \dot{G}$. Let $\psi \circ \varphi^{-1}$ be induced by a self-homeomorphism f of U/G. By using Theorem 2 of Kra [6] once again, we see that an essential hyperbolic element g of G determines a hyperbolic modular transformation $\varphi \circ g \circ \varphi^{-1}$ which is, of course, induced by a irreducible self-homeomorphism f_0 on $U/\dot{G}$ (Theorem 6 of Baers [4]). f_0 is irreducible if and only if $f \circ f_0 \circ f^{-1}$ is irreducible. It follows that $\psi \circ g \circ \psi^{-1} \in \operatorname{Mod} \dot{G}$ is hyperbolic. Similarly, a self-homeomorphism s of $U/\dot{G}$ is a spin about $\hat{a}$ (that is, $s = h_{C_2} \circ h_{C_1}^{-1}$, where h_{C_i} is the Dehn twist about C_i, and C_1 and C_2 bound a cylinder which contains the only puncture $\hat{a}$) if and only if $f \circ s \circ f^{-1}$ is a spin about $f(\hat{a})$. More precisely, we see that $f \circ s \circ f^{-1}$ is isotopic to $h_{f(C_2)} \circ h_{f(C_1)}^{-1}$. Furthermore, C_1 bounds a punctured disk if and only if $f(C_1)$ bounds a punctured disk. Hence, the argument in the previous section carries over word by word for this general case.

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