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THE NONCOMMUTATIVITY OF HECKE ALGEBRAS ASSOCIATED TO WEYL GROUPS

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ABSTRACT. We prove that the Hecke algebra $\mathscr{H}(W, W_J)$, where W is a Weyl group of spherical type and W_J is a standard parabolic subgroup of W of corank ≥ 2, is noncommutative.

1. INTRODUCTION

Let G be a finite group, H a subgroup of G, and K a field of characteristic zero. Let $\mathscr{H}(G, H)$ be the Hecke algebra of G with respect to H, defined over K. It may be viewed as the algebra of double cosets of H in G (see [K] for details) or, equivalently, as the subalgebra $e_H K[G] e_H$ of $K[G]$, where $e_H = (1/\#H) \sum_{h \in H} h$.

A natural question to ask about $\mathscr{H}$ is whether it is commutative. When G is a Weyl group, Iwahori gave the following theorem that tells when $\mathscr{H}$ is commutative:

Theorem 1.1 ([I, Theorem 2]). *Let W be a Weyl group and let W_J be a parabolic subgroup of W. $\mathscr{H}(W, W_J)$ is commutative if and only if $W_J w W_J = W_J w^{-1} W_J$ for all $w \in W$.*

Let W be a Weyl group of spherical type. It has been known for some time now which maximal parabolic subgroups W_J of W give rise to commutative Hecke algebras (see, for example, Theorem 10.4.11 of [BCN]). In this paper, we examine the case where W_J is a nonmaximal parabolic subgroup of W and prove the following:

Theorem 1.2. *Let (W, R) be a Weyl group of type B_n, D_n, E_6, E_7, E_8, F_4 or G_2 with generating set $R = \{r_1, r_2, \ldots, r_n\}$. Let $J \subset R$. If $\#(R \backslash J) \geq 2$, then $\mathscr{H}(W, W_J)$ is noncommutative.*

That $\mathscr{H}(W, W_J)$, where W is of type A_n and W_J is a nonmaximal parabolic, is noncommutative is already known (see Lemma III.3.5 of [K]), so we will not consider it.

Write the Coxeter diagrams as on [Su, p. 306] and number the generators from left to right.

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If W is a Weyl group with generators $R = \{r_1, r_2, \dots, r_n\}$, and $J \subset R$, we say W_J has *corank* k if $\#\{R\backslash J) = k$. We denote by P_i the parabolic subgroup of W generated by $R\backslash\{r_i\}$, by P_{ij} the parabolic subgroup of W generated by $R\backslash\{r_i, r_j\}$, etc.

We begin by introducing data about $\mathscr{H}(W, W_J)$ for various W and W_J, derived using CAYLEY software. From this data, and Theorems 2.1 and 3.2, we construct the proof of Theorem 1.2. In the following tables, *Dimension* refers to the dimension of the given Hecke algebra, while *Dimension of invariants* refers to the dimension of the set of elements of $\mathscr{H}(W, W_J)$ which are fixed by the canonical involution. The column w gives an element of W for which $W_J w W_J \neq W_J w^{-1} W_J$.

What Theorem 1.1 means for us is that if the numbers in the *Dimension* and *Dimension of invariants* columns are different, then $\mathscr{H}(W, W_J)$ is noncommutative.

2. The exceptional Weyl groups

Table 1 lists data for selected Hecke algebras of corank 2 subgroups in F_4, E_6, E_7 and E_8, computed using CAYLEY. Those cases not listed in Table 1 are noncommutative, too. $\mathscr{H}(G_2, P_{12})$ is noncommutative because $P_{12} = \langle e \rangle$, so $\mathscr{H}(G_2, P_{12}) = K[G_2]$. The others arise from subgroups that are contained in subgroups that give rise to noncommutative Hecke algebras (see Theorem 10.4.11 of [BCN] or Table 1 of [A]), so they also are noncommutative by the following theorem, which also implies that all Hecke algebras of subgroups in F_4, E_6, E_7 and E_8 of corank ≥ 3 are noncommutative.

Theorem 2.1. *Let G be a finite group. Let S, T be subgroups of G with $T \subset S$. If $\mathscr{H}(G, S)$ is noncommutative, then $\mathscr{H}(G, T)$ is noncommutative.*

Proof. Let us view $\mathscr{H}(G, S)$ and $\mathscr{H}(G, T)$ as $e_S K[G] e_S$ and $e_T K[G] e_T$. Let $\{T_i\}$ be the standard basis of $e_T K[G] e_T$. By Lemma 2.3 of [CIK], we can form a basis $\{Z_j\}$ of $e_S K[G] e_S$ by summing elements of $\{T_i\}$, i.e.,

$$Z_j = \sum_{i \in J_j} T_i,$$

for some index set J_j.

Suppose $\mathscr{H}(G, T) (\cong e_T K[G] e_T)$ is commutative. Then for all i_1 and i_2, $T_{i_1} \cdot T_{i_2} = T_{i_2} \cdot T_{i_1}$. Hence, for any j_1 and j_2,

$$\begin{aligned} Z_{j_1} \cdot Z_{j_2} &= \left(\sum_{i \in J_{j_1}} T_i \right) \left(\sum_{i \in J_{j_2}} T_i \right) = \sum_{\substack{i \in J_{j_1} \\ j \in J_{j_2}}} T_i \cdot T_j \\ &= \sum_{\substack{i \in J_{j_1} \\ j \in J_{j_2}}} T_j \cdot T_i = \left(\sum_{i \in J_{j_2}} T_i \right) \left(\sum_{i \in J_{j_1}} T_i \right) = Z_{j_2} \cdot Z_{j_1}. \end{aligned}$$

Thus, $\mathscr{H}(G, S) (\cong e_S K[G] e_S)$ is commutative. □

TABLE 1. Hecke algebras relative to corank 2 subgroups of F_4, E_6, E_7, E_8.

Hecke algebra	Dimension	Dimension of invariants	w
$\mathscr{H}(F_4, P_{14})$	33	25	$r_1r_2r_3r_4$
$\mathscr{H}(E_6, P_{14})$	26	20	$r_1r_2r_3r_4$
$\mathscr{H}(E_6, P_{16})$	21	16	$r_1r_2r_3r_5r_6$
$\mathscr{H}(E_6, P_{46})$	26	20	$r_4r_3r_5r_6r_4$
$\mathscr{H}(E_7, P_{15})$	70	52	$r_1r_2r_3r_4r_5$
$\mathscr{H}(E_7, P_{17})$	36	28	$r_1r_2r_3r_4r_6r_7$
$\mathscr{H}(E_7, P_{57})$	70	52	$r_5r_4r_6r_7r_5$
$\mathscr{H}(E_8, P_{18})$	114	80	$r_1r_2r_3r_4r_5r_7r_8$

Note. One may also use Theorem 3.2 to show the noncommutativity of all of the Hecke algebras listed in Table 1, except for $\mathscr{H}(F_4, P_{14})$.

3. D_n

The next few results show that for all n, and for all i, j with $1 \leq i < j \leq n$, $\mathscr{H}(D_n, P_{ij})$ is noncommutative.

Proposition 3.1. *Let (W, R) be a Coxeter system of finite rank. Let $I, J \subset R$, and let $w \in W_J$. Then $W_I w W_I \cap W_J = (W_I \cap W_J)w(W_I \cap W_J)$.*

Proof. Analogous to the proof of Corollary 12.8 of [T]. □

Theorem 3.2. *Let (W, R) and (W', R') be Weyl groups such that $R' \subseteq R$. Let $J' \subseteq R'$. Let $J = J' \cup (R \backslash R')$. If $\mathscr{H}(W', W'_{J'})$ is noncommutative, then so is $\mathscr{H}(W, W_J)$.*

Proof. Suppose $\mathscr{H}(W, W_J)$ is commutative. Then by Theorem 1.1, $W_J w W_J = W_J w^{-1} W_J$ for all $w \in W$.

Let $w \in W'$. By Proposition 3.1,

$$\begin{aligned} W'_{J'} w W'_{J'} &= W_J w W_J \cap W' \\ &= W_J w^{-1} W_J \cap W' \\ &= W'_{J'} w^{-1} W'_{J'}. \end{aligned}$$

Hence, by Theorem 1.1, $\mathscr{H}(W', W'_{J'})$ is commutative. □

We now continue our analysis of D_n.

Let $n \geq 4$. We wish to show that $\mathscr{H}(D_n, P_{ij})$ is noncommutative for all $1 \leq i < j \leq n$. Let $i, j \in \{1, 2, \dots, n\}$, $i \neq j$. At least one of $\{r_1, r_{n-1}, r_n\}$ is in $R \backslash \{r_i, r_j\}$. We have two cases to consider:

Case 1. r_{n-1} or r_n in $R\backslash\{r_i, r_j\}$.

Let W' be the group generated by R', where $R' = R\backslash\{r_{n-1}\}$ or $R' = R\backslash\{r_n\}$. W' is isomorphic to A_{n-1}.

Let P'_{ij} be the parabolic subgroup of W' generated by $R'\backslash\{r_i, r_j\}$. By Lemma III.3.5 of [K], $\mathscr{H}(W', P'_{ij})$ is noncommutative. Hence, by Theorem 3.2, $\mathscr{H}(D_n, P_{ij})$ is noncommutative.

Case 2. $i = n-1$, $j = n$.

We will induct on n, beginning with $n = 4$. For $n = 4$, let $W' = \langle r_2, r_3, r_4\rangle$. W' is isomorphic to A_3. Let $P' = \langle r_2\rangle$. By Lemma III.3.5 of [K], $\mathscr{H}(W', P')$ is noncommutative, so by Theorem 3.2, $\mathscr{H}(D_4, P_{34})$ is noncommutative.

Suppose that $\mathscr{H}(D_k, P_{k-1,k})$ is noncommutative for all $4 \le k < n$. Let W' be generated by $R' = \{r_2, r_3, \dots, r_n\}$, and P' be generated by $R'\backslash\{r_{n-1}, r_n\}$. W' is isomorphic to D_{n-1}, so by induction, $\mathscr{H}(W', P')$ is noncommutative. Hence, by Theorem 3.2, $\mathscr{H}(D_n, P_{n-1,n})$ is noncommutative. □

4. B_n

We now turn to B_n. Table 2 contains data about Hecke algebras with respect to corank 2 subgroups of B_3 and B_4.

As before, we see that these corank 2 parabolics give rise to noncommutative Hecke algebras. We also know that $\mathscr{H}(B_2, P_{12})$ is noncommutative, since $P_{12} = \langle e\rangle$, so

$$\mathscr{H}(B_2, P_{12}) \cong K[B_2],$$

which is noncommutative. We will now prove that $\mathscr{H}(B_n, P_{ij})$ is noncommutative for all $n \ge 2$ and all parabolic subgroups of the form P_{ij}, where $i < j$.

TABLE 2. Hecke algebras relative to corank 2 subgroups of B_3, B_4.

Hecke algebra	Dimension	Dimension of invariants	w
$\mathscr{H}(B_3, P_{12})$	16	13	r_1r_2
$\mathscr{H}(B_3, P_{13})$	14	12	$r_1r_2r_3$
$\mathscr{H}(B_3, P_{23})$	14	12	r_2r_3
$\mathscr{H}(B_4, P_{12})$	17	14	r_1r_2
$\mathscr{H}(B_4, P_{13})$	33	26	$r_1r_2r_3$
$\mathscr{H}(B_4, P_{14})$	20	17	$r_1r_2r_3r_4$
$\mathscr{H}(B_4, P_{23})$	33	26	r_2r_3
$\mathscr{H}(B_4, P_{24})$	34	27	$r_2r_3r_4$
$\mathscr{H}(B_4, P_{34})$	20	17	r_3r_4

Let W' be generated by $R' = R\backslash\{r_n\}$. W' is isomorphic to A_{n-1}. Let P'_{ij} be generated by $R'\backslash\{r_i, r_j\}$. By Lemma III.3.5 of [K], $\mathscr{H}(W', P'_{ij})$ is noncommutative, so by Theorem 3.2, $\mathscr{H}(B_n, P_{ij})$ is noncommutative.

The remaining case to consider is $\mathscr{H}(B_n, P_{in})$, $i < n$. Suppose that $i > 1$. If we let W' be generated by $R' = R\backslash\{r_1, r_2, \ldots, r_{i-1}\}$, then by Theorem 3.2, we would just need to show that $\mathscr{H}(B_n, P_{1n})$ is noncommutative for all $n \geq 2$. However, if we try to use Theorem 3.2 to do this, we run into the following problem: Let $n > 4$. Let $1 < k < n$. Let W' be generated by $R' = R\backslash\{r_k\}$. Then W' is isomorphic either to $A_{k-1} \times B_{n-k}$ or to $A_{n-2} \times A_1$, depending on k. Let P' be generated by $R'\backslash\{r_1, r_n\}$, so P' is isomorphic either to $A_{k-2} \times A_{n-k-1}$ or to $A_{n-3} \times \langle e \rangle$, respectively. By Theorem I.6.3 of [K], $\mathscr{H}(W', P')$ is isomorphic either to $\mathscr{H}(A_{k-1}, A_{k-2}) \otimes \mathscr{H}(B_{n-k}, A_{n-k-1})$ or to $\mathscr{H}(A_{n-2}, A_{n-3}) \otimes \mathscr{H}(A_1, \langle e \rangle)$. In either case, by Proposition III.3.3 of [K] and [CIK], both factors of the tensor product are commutative, so $\mathscr{H}(W', P')$ is commutative. But Theorem 3.2 can be used only if $\mathscr{H}(W', P')$ is noncommutative. Thus, we need another approach. A closer look at the root system for B_n will provide the needed tools.

Proposition 4.1. *Let $n \geq 2$. $\mathscr{H}(B_n, P_{1n})$ is noncommutative.*

Proof. Let $\{\varepsilon_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq n-1$ and let $\alpha_n = \varepsilon_n$. By Section 12.1 of [Hu], $\{\alpha_i\}_{i=1}^n$ forms a base of the root system of type B_n. By Theorem 10.3 of [Hu], the standard reflections $\sigma_i = \sigma_{\alpha_i}$, $i = 1, 2, \ldots, n$, generate the Weyl group W of type B_n, where for $1 \leq i \leq n-1$, σ_i acts by permuting ε_i and ε_{i+1} while $\sigma_n(\varepsilon_n) = -\varepsilon_n$. Thus, everything in W permutes and/or changes the signs of the ε_i, for $i = 1, 2, \ldots, n$. Let $w = \sigma_1\sigma_2\cdots\sigma_n$ (the Coxeter element). By an easy calculation (working right-to-left), $w(\varepsilon_1) = \varepsilon_2$, while $w^{-1}(\varepsilon_1) = -\varepsilon_n$.

Let P_{1n} be the parabolic subgroup of W generated by $\{\sigma_2, \ldots, \sigma_{n-1}\}$. Let $w_1, w_2 \in P_{1n}$. Since $w_2 \in P_{1n}$, it is the product of elements of $\{\sigma_2, \ldots, \sigma_{n-1}\}$, so it fixes ε_1, i.e., $w_2(\varepsilon_1) = \varepsilon_1$. Hence, $ww_2(\varepsilon_1) = w(\varepsilon_1) = \varepsilon_2$.

Since $w_1 \in P_{1n}$, σ_n does not appear in its decomposition, so $w_1(\varepsilon_i) = \varepsilon_j$, $\forall i = 1, 2, \ldots, n$ and some $j \in \{1, 2, \ldots, n\}$. The important point here is that ε_j is a positive linear combination of the fundamental roots. In particular, $w_1(\varepsilon_2)$ will be a positive linear combination of fundamental roots, so $w_1ww_2(\varepsilon_1) = w_1(\varepsilon_2)$ is also one. On the other hand, we know $w^{-1}(\varepsilon_1) = -\varepsilon_n$, i.e., $w^{-1}(\varepsilon_1)$ is not one. Therefore,

$$w_1ww_2 \neq w^{-1} \quad \forall w_1, w_2 \in P_{1n}.$$

Hence, $P_{1n}wP_{1n} \neq P_{1n}w^{-1}P_{1n}$, so by Theorem 1.1, $\mathscr{H}(B_n, P_{1n})$ is noncommutative. □

As for the parabolic subgroups of corank > 2, since they all occur as subgroups of parabolics of corank 2, Theorem 2.1 will imply that they also all give rise to noncommutative Hecke algebras.

Note. These results have some interest for the representation theory of Weyl groups. As is well known, $\mathscr{H}(W, W_J)$ is the centralizer algebra of the induced representation $1_{W_J}^W$, so the commutativity of $\mathscr{H}$ tells whether this representation is multiplicity-free. Hence, one way to express our results is to say that for any classical Weyl group W and parabolic subgroup P of corank ≥ 2,

1_P^W is not multiplicity-free. This is also the point of interest for the theory of distance-regular graphs.

Note. The data used in compiling Tables 1 and 2 are available from the author. For a discussion of the algorithms used, see [A].

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