CONVEX FUNCTIONS AND SCHWARZ DERIVATIVES

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Abstract. If the lower Schwarz derivate of a continuous function is nonnegative, then it is convex. The main result in this paper is that if the lower Schwarz derivate of a measurable function \( f \) is nonnegative, then there is a dense open set with \( f \) convex on each component.

We begin by recalling the definition of a lower Schwarz derivate.

**Definition 1.** Let \( f \) be a function defined on an open interval \( I \), and \( x \in I \). For each \( h > 0 \) so small that \( x + h \) and \( x - h \) both lie in \( I \) define

\[
\Delta_2 f(x, h) = \frac{(f(x+h)+f(x-h)-2f(x))}{h^2},
\]

and let

\[
LD_2 f(x) = \liminf_{h \to 0^+} \Delta_2 f(x, h).
\]

\( LD_2 \) is called the lower Schwarz derivate.

It is well known and easy to prove that if \( f \) is continuous on an open interval \( I \) and if the lower Schwarz derivate (even the upper Schwarz derivate) is nonnegative, then \( f \) is convex on \( I \). For this and some other simple properties of convex functions the reader is referred to [3].

A natural question to ask is whether \( LD_2 f(x) > 0 \) still implies convexity on \( I \) when continuity is replaced by weaker conditions. The first major result in this direction is the following result of C. Weil. (See [2].)

**Theorem 2.** Let \( I \) be an open interval and \( f \) a Baire one, Darboux function defined on \( I \). If \( LD_2 f(x) > 0 \) for each \( x \in I \), then \( f \) is convex on \( I \).

Later, Z. Buczolich proved the following theorem. (See [1].)

**Theorem 3.** Let \( I \) be an open interval and \( f \) a measurable function defined on \( I \) such that \( LD_2 f(x) \geq 0 \) for each \( x \in I \). If \( f \) is a sum of a continuous function and Darboux function, then \( f \) is convex on \( I \).

Weil's proof of Theorem 2 is lengthy and very complicated. Buczolich's proof of Theorem 3 shows how to modify Weil's proof for measurable functions and it is based on the following property of measurable functions with positive Schwarz derivate.
Lemma 4. Let $f$ be a real-valued measurable function defined on an open interval $I$ such that $DL_2 f(x) > 0$ for every $x \in I$. If $f$ is bounded by $B$ on a dense set of some open interval $K$, then $f$ is bounded by $B$ on all of $K$.

We give a proof of this lemma that differs from the proof in [1]. For the proof of Lemma 4, Lemma 5, Theorem 6, and Corollary 7 below, we define for each $x \in I$

$$
\delta(x) = \sup\{\gamma > 0: f(x + h) + f(x - h) - 2f(x) > 0 \text{ for all } 0 < h < \gamma\}.
$$

For $n \in \mathbb{N}$, let

$$
A_n = \{x: \delta(x) \geq 1/n \text{ and } f(x) \leq n\}.
$$

If $DL_2 f(x) > 0$ for all $x \in I$, then $\delta(x)$ is well defined and $\bigcup_{n=1}^{\infty} A_n = I$.

Proof of Lemma 4. Let $A = \{x \in K: f(x) \leq B\}$. If $A$ has full measure in $K$ let $x$ be any point in $K$. Then one can find $0 < h < \delta(x)$ such that both $x - h$ and $x + h$ are in $A$, which implies that $f(x) \leq B$, and we are done.

If $A$ does not have full measure, then for some positive rational number $m$,

$$
C_m = \{x \in K: B + m < f(x) < B + 2m\}
$$

has positive measure, and then for some $n$, $C_m \cap A_n$ has positive outer measure. Then there is an interval $[a, b] \subset K$, such that $b - a < 1/n$ and where the outer measure of $C_m \cap A_n$ is bigger than $\frac{1}{6}(b - a)$. Let $J_0 = [a, a + \frac{b - a}{3}]$. Then $D = J_0 \cap C_m \cap A_n$ has outer measure bigger than $\frac{1}{6}(b - a)$. Choose a point $x \in A$ such that $x < a$ and $a - x < (b - a)/3$. Then the set $2D - x$ has outer measure bigger than $\frac{1}{3}(b - a)$. If $d \in D$, then $f(x) - 2f(d) + f(2d - x) > 0$. Hence $f(2d - x) > 2(B + m) - B = B + 2m$. Then the disjoint sets $f^{-1}((B + 2m, \infty)) \cap [a, b]$ and $f^{-1}((-\infty, B + 2m)) \cap [a, b]$ are measurable, and since $2D - x \subset f^{-1}((B + 2m, \infty)) \cap [a, b]$ and $D \subset f^{-1}((-\infty, B + 2m)) \cap [a, b]$, their measures are bigger than $\frac{1}{6}(b - a)$ and $\frac{1}{3}(b - a)$ respectively, a contradiction. $\square$

That the same conclusion holds if measurability in Lemma 4 is replaced by Baire property is the statement of Lemma 5.

Lemma 5. Let $f$ be a real-valued function with the property of Baire defined on an open interval $I$ such that $DL_2 f(x) > 0$ for every $x \in I$. If $f$ is bounded by $B$ on a dense set of some open interval $K$, then $f$ is bounded by $B$ on all of $K$.

Proof. Let $A = \{x \in K: f(x) \leq B\}$. If $A$ is a residual set in $K$, then for any $x \in K$ we can pick $0 < h < \delta(x)$ such that both $x - h$ and $x + h$ are in $A$, which implies that $f(x) \leq B$, and we are done.

If $A$ is not residual, then for some positive rational number $m$,

$$
C_m = \{x \in K: B + m < f(x) < B + 2m\} \neq \emptyset
$$

is second category in $K$. Since $f$ has the property of Baire, there is a subinterval $[c, d] \subset K$ such that $C_m$ is residual in $[c, d]$. By the Baire Category Theorem there is an interval $[a, b] \subset [c, \frac{c + d}{3}]$ such that for some integer $n$, $C_m \cap A_n$ is second category in $[a, b]$. Without loss of generality we may assume that $b - a < \frac{1}{2n}$. Since $A$ is dense in $K$, one can find a point $x \in A$ such that
$x < a$ and $a - x < \frac{1}{b_n}$. The set $2(C_m \cap A_n \cap [a, b]) - x$ is second category in $[c, d]$. Therefore the intersection $(2(C_m \cap A_n \cap [a, b]) - x) \cap (C_m \cap [c, d])$ is nonempty. Let $y$ be from the intersection. Then there is a $z \in C_m \cap A_n$ such that $z = \frac{a + b}{2}$. Hence $f(x) - 2f(z) + f(y) > 0$. It follows that $f(x) > 2f(z) - f(y) > 2(B + m) - B + 2m = B$, which contradicts $x \in A$. □

The main result of the paper is the following theorem.

**Theorem 6.** Let $I$ be an open interval. If $f$ is a measurable function (or with the property of Baire) defined on $I$ such that $DL_2f(x) \geq 0$ for every $x \in I$, then there is an open dense set $G$ in $I$ such that $f$ is convex on every component of $G$. Conversely, if $G$ is an open dense set in $I$, then there is a measurable function $f$ (with the property of Baire) defined on $I$ such that $DL_2f(x) \geq 0$ on $I$ and if $J \subset I$ is any interval on which $f$ is convex, then $J \subset G$.

**Proof.** In the first part of Theorem 6 it suffices to prove that for a measurable function (or with the property of Baire) with $DL_2f(x) > 0$ on $I$, there is a dense open set $G$ in $I$ such that $f$ is convex on each component of $G$. To see this, consider $F(x) = f(x) + x^2$. Then $DL_2F(x) > 0$ on $I$. By proving the theorem for $F$, we actually get that $F$ is continuous on each component of $G$. Then $f$ is continuous on every component of $G$ and since $DL_2f(x) \geq 0$, $f$ is convex on every component of $G$. So, we assume $DL_2f(x) > 0$ for all $x \in I$.

For each $a, b \in I$, define a function $g$ by subtracting from $f$ the line connecting $(a, f(a))$ and $(b, f(b))$, i.e.,

\[(2) \quad g(x) = f(x) - \frac{x-a}{b-a} f(b) - \frac{b-x}{b-a} f(a).\]

Then $g(a) = g(b) = 0$ and

\[(3) \quad g(x + h) + g(x - h) - 2g(x) = f(x + h) + f(x - h) - 2f(x) \quad \text{for all } x \in I.\]

Let $K \subset I$ be an interval. Since $K \subset \bigcup_{n=1}^{\infty} A_n$, by the Baire Category Theorem there is an integer $n$ and an interval $J \subset K$ of length less than $1/n$ so that $A_n$ is dense in $J$. Let $[a, b] \subset J$, $g$ be as in (2), and $c = \sup\{g(x): x \in [a, b]\}$. By Lemma 4 (Lemma 5), $c < +\infty$. Suppose that $c > 0$. This will lead to a contradiction as follows.

First we show that there is a sequence $\{y_k\} \subset A_n \cap (a, b)$ such that $g(y_k)$ converges to $c$. Suppose not. Then there is an $\varepsilon > 0$ such that $\sup\{g(y): y \in A_n\} < c - \varepsilon$. Pick an $x \in (a, b)$ such that $g(x) > c - \varepsilon/2$. Since $A_n$ is dense in $[a, b]$, there is a $y \in A_n \cap (a, b)$ such that $|y - x| < \delta(x)$ and $2x - y \in (a, b)$. Therefore, by (1) and (3) $g(x) < 1/2(g(y) + g(2x - y)) < 1/2(c - \varepsilon + c) = c - \varepsilon/2$, a contradiction.

Since $\{y_k\} \subset [a, b]$, we may assume it is convergent. Let $y \in [a, b]$ be the limit point of the sequence. If $g(y) = c > 0$, then $y \in (a, b)$, and if $z \in (a, b)$ is such that $|z - y| < \delta(y)$ and $2y - z \in (a, b)$, then (1) and (3) imply that either $g(z)$ or $g(2z - y)$ has to be strictly bigger than $g(y)$, which violates the choice of $c$. Therefore $g(y) < c$. Now pick $k$ so that $c - g(y_k) < g(y_k) - g(y)$ and that $2y_k - y \in (a, b)$. Then, again by (3), and the fact that $\delta(y_k) \geq 1/n$, we have $g(2y_k - y) > 2g(y_k) - g(y) > c + g(y) - g(y) = c$, a contradiction. Therefore,

\[(4) \quad \text{if } x \in [a, b] \subset J, \quad \text{then } g(x) \leq 0.\]
This implies that for every $x \in [a, b]$, $f(x)$ is below the line segment connecting the points $(a, f(a))$ and $(b, f(b))$. Since $[a, b]$ was an arbitrary interval in $J$, we have that $f$ is convex (and thus continuous) on $J$. Let $G \subset I$ be the union of all open intervals on which $f$ is convex. If $(a, b)$ is a component interval of $G$, then $f$ is continuous on $(a, b)$, and since $DL_2 f(x) > 0$ for all $x \in (a, b)$, it is convex there. Finally, since $K$ was an arbitrary open interval in $I$, $G$ is dense in $I$.

Conversely, let $G$ be an open dense subset of $I$. Then

$$f(x) = \begin{cases} 
  x^2 & \text{if } x \in G, \\
  0 & \text{otherwise}
\end{cases}$$

satisfies the conditions of the theorem. □

The following corollary is a generalization of Theorem 3.

**Corollary 7.** Let $I$ be an open interval and $f$ a measurable function (or with the property of Baire) defined on $I$ with $DL_2 f(x) \geq 0$ for each $x \in I$. Let $G$ be the dense open set from Theorem 6. If $f$ is continuous on the closure of every component of $G$, then $f$ is convex on $I$.

**Proof.** Let $C = I \setminus G$. Note that $G$ is a maximal open set such that $f$ is convex on every component of $G$. If $b \in C$ is an isolated point, then there are intervals $(a, b)$ and $(b, d)$ in $G$. Then $f$ is continuous at $b$ and hence $f$ is convex on $(a, d)$. Therefore $C$ is perfect.

If $C \neq \emptyset$, then by the Baire Category Theorem there is an integer $n$ and an open interval $J \subset I$ of length less than $1/n$ so that $A_n$ is dense in $C \cap J \neq \emptyset$. Let $[a, b] \subset J$ and $g$ be as in (2). Our goal is to show that if $x \in [a, b]$, then $g(x) \leq 0$. If $[a, b] \subset G$, then, since in that case $g$ is convex on $[a, b]$, for every $x \in [a, b]$ we have that $g(x) \leq 0$. So we may assume $[a, b] \cap A_n \neq \emptyset$. Let $c = \sup \{g(x) : x \in A_n \cap [a, b]\}$. Let $(d, e)$ be a component of $G \cap [a, b]$.

If $e = b$ and $g(d) \leq g(e)$, then by convexity $g(x) \leq g(b) = 0$ for all $x \in [d, e]$. If $e < b$ and $g(d) \leq g(e)$, then for $x \in (e, 2e - d), x - e < \delta(e)$, $g(2e - x) - 2g(e) + g(x) > 0$. Since $d < 2e - x < e$, it follows that $g(x) > g(e)$. In particular, this is true for some $x \in A_n$ since $A_n$ is dense in $C \cap [a, b]$. We conclude that if $y \in [d, e]$, then $g(y) \leq \max(g(x), 0) \leq |c|$. A similar argument holds if $g(d) \geq g(e)$.

Now pick any $x \in [a, b]$. Then for some $0 < h < \delta(x)$, both $x - h$ and $x + h$ are in $G \cap [a, b]$. Since $g(x + h)$ and $g(x - h)$ are bounded by $|c|$, it follows that $g(x) \leq |c|$.

Assume $c > 0$. Let $\{y_k\} \subset [a, b] \cap A_n$ be a convergent sequence such that $g(y_k) \to c$ and let $y \in [a, b]$ be the limit point of $\{y_k\}$. Proceeding as in the proof of (4), we get $g(y) < c$ which gives a nearby point $z$ such that $g(z) > c$, which is a contradiction.

Therefore $c = 0$, and hence if $x \in [a, b]$, then $g(x) \leq 0$. As in the proof of Theorem 6, this implies $f$ never goes above the line segment joining $(a, f(a))$ and $(b, f(b))$. Since $[a, b]$ is arbitrary, $f$ is convex on $J$, contradicting the maximality of $G$. Hence $C = \emptyset$. □

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