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# BOUNDS FOR THE BETTI NUMBERS OF GENERALIZED COHEN-MACAULAY IDEALS

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ABSTRACT. Upper bounds for the Betti numbers of generalized Cohen-Macaulay ideals are given. In particular, for the case of non-degenerate, reduced and irreducible projective curves we get an upper bound which only depends on their degree.

## 0. INTRODUCTION

Let $I$ be a homogeneous ideal of a polynomial ring $S = \mathbf{K}[x_1, \dots, x_n]$ over a field $\mathbf{K}$, $R = S/I$, $\underline{M} := (x_1, \dots, x_n)$, $\underline{m} = \underline{M}R$ and $e = e(I) := e(R)$ the multiplicity of $R/I$. $I$ is said to have a property $P$ if $R$ has this property $P$.

It is a classical question to give upper and lower bounds for the Betti numbers, $\beta_i$, of $S/I$. A well-known conjecture due to Buchsbaum-Eisenbud says that $\beta_i(S/I) \geq \binom{n}{i}$ for 0-dimensional ideals, and very recently Valla has given sharp upper bounds for the case of C.M. ideals (see [V]). The goal of this paper is to extend Valla's result to generalized C.M. ideals, i.e. ideals whose local cohomology modules $H^i_{\underline{m}}(R)$ are of finite length for all $i < \dim(R)$. As in [H], the key point is to reduce the computation to the case of C.M. ideals.

Now we give a brief description of the paper. In §1, we fix notations and recall some results needed later on. In §2, in order to prove our main result (Theorem 2.6), we first reduce to the case of 0-dimensional ideals and then we extend Valla's bounds to arbitrary (not necessarily non-degenerate) 0-dimensional ideals. As a consequence and related to Buchsbaum-Eisenbud's conjecture we get the upper bound $\beta_i(S/I) \leq \binom{n}{i}e$ for the Betti numbers of any homogeneous 0-dimensional ideal $I$. In the last section, applying our results we obtain upper bounds for the Betti numbers of the homogeneous ideal of some special projective schemes. In particular, for the case of non-degenerate, reduced and irreducible projective curves, $C$, we get an upper bound which only depends on the degree of the curve $C$.

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## 1. Notation and preliminaries

Throughout this paper we make the following conventions:

$$\binom{m}{0} = 1 \quad \text{if } m \geq 0, \qquad \text{and} \qquad \binom{m}{k} = 0 \quad \text{if } m < k.$$

The following combinatorial formulae will be useful to us:

$$\binom{n}{i} + \cdots + \binom{n-l}{i} = \binom{n+1}{i+1} - \binom{n-l}{i+1}, \tag{1.1}$$

$$\sum_{i=a}^{b} \binom{i}{a}\binom{c+i-1}{i} = \binom{c+a-1}{a}\binom{c+b}{c+a}, \tag{1.2}$$

$$\sum_{i=a}^{b} i\binom{i}{a} = -\binom{b+2}{a+2} + (b+1)\binom{b+1}{a+1}. \tag{1.3}$$

If $m$ and $i$ are positive integers, then $m$ can be written uniquely in the form

$$m = \binom{m(i)}{i} + \binom{m(i-1)}{i-1} + \cdots + \binom{m(j)}{j},$$

where $m(i) > m(i-1) > \cdots > m(j) \geq j \geq 1$. This is called the $i$-binomial expansion of $m$. We let

$$m^{\langle i \rangle} = \binom{m(i)+1}{i+1} + \binom{m(i-1)+1}{i} + \cdots + \binom{m(j)+1}{j+1},$$
$$m_{\langle i \rangle} = \binom{m(i)-1}{i} + \binom{m(i-1)-1}{i-1} + \cdots + \binom{m(j)-1}{j}$$

and $0^{\langle i \rangle} = 0$. We define $r_{\langle t \rangle (0)} := r$ and inductively $r_{\langle t \rangle (k)} = (r_{\langle t \rangle (k-1)})_{\langle t \rangle}$.

If $I$ is a 0-dimensional ideal we denote its Hilbert function by $H_{S/I}$.

(1.4) Following [ERV], §4, we denote by $J(e, h)$ the unique 0-dimensional lex-segment ideal in $S = \mathbf{K}[x_1, \ldots, x_h]$ with the Hilbert function $H_{S/J(e,h)} = (1, h, \binom{h+1}{2}, \ldots, \binom{h+t-2}{t-1}, r, 0, \ldots)$, where $t = t(e, h)$ is the unique integer such that $\binom{h+t-1}{t-1} \leq e < \binom{h+t}{t}$ and $r = r(e, h) = e - \binom{h+t-1}{t-1}$. We set $t(e, 0) := 1$.

(1.5) For $p = 0, \ldots, h-1$, denote by $J_p(e, h)$ (resp. $I_p$) the image of $J(e, h)$ (resp. $I$) in the polynomial ring $S_p := \mathbf{K}[x_1, \ldots, x_p]$ under the canonical projection.

For short we also use the notation $e_p(e, h) = e(S_p/J_p(e, h))$. In particular, $e_0(e, h) = 1$. By [V], $e_q(e, n) = \binom{t-1+q}{q} + r_{\langle t \rangle (n-q)}$ for all $0 \leq q \leq n-1$.

(1.6) Let $H = (1, H(1), \ldots, H(a), \ldots)$ and $L = (1, L(1), \ldots, L(b), \ldots)$ be the Hilbert functions of two 0-dimensional homogeneous ideals of some polynomials rings, where $H(a) \neq 0$ and $L(b) \neq 0$. We write $H\cdot \geq L$ if $H(i) \geq L(i)$ for $i = 0, \ldots, a-1$.

(1.7) By [ERV], Corollary 2.8, $H_{S_{h-1}/J_{h-1}(e,h)}(n) = (H_{S/J(e,h)}(n))_{\langle n \rangle}$ for all $n \geq 1$.

The following lemma will be useful to us and it is essentially contained in the proof of [ERV], Theorem 3.10.

**Lemma 1.8.** *Assume that* $1 \leq h \leq h'$. *Then, for all* $p \geq 1$, *we have* $e_{h-p}(e, h) \leq e_{h'-p}(e, h')$.

*Proof.* Since $e(e, h-p) = 0$ for $p \geq h$, we have only to consider the case $1 \leq p \leq h-1$. Let $S = \mathbf{K}[x_1, \dots, x_h]$ and $S' = \mathbf{K}[x_1, \dots, x_{h'}]$. By the definition of $t$ and $t'$ in (1.4) we have $t' = t'(e, h') \leq t = t(e, h)$. Hence, by (1.4) and (1.6), $H_{S'/J(e,h')} \geq H_{S/J(e,h)}$. Moreover $e(S'/J(e, h')) = e(S/J(e, h)) = e$. By repeated application of [ERV], Lemma 3.9, and (1.7) we get $e_{h-p}(e, h) \leq e_{h'-p}(e, h')$, as required.

(1.9) (See [V], Proposition 2.) Let $I$ be a 0-dimensional non-degenerated lex-segment ideal of $S$. Then, for every $i = 1, \dots, n$ we have

$$\beta_i(S/I) = \sum_{p=i-1}^{n-1} \binom{p}{i-1} e(S_p/I_p).$$

(1.10) For the basic properties of Buchsbaum as well as generalized C.M. rings we refer the reader to the book [SV]. A ring $R$ is called a generalized C.M. ring if the length of the local cohomology modules $H^i_{\mathfrak{m}}(R)$ is finite for all $i < \dim(R)$. In this case we set $I(R) := \bigoplus_{i=0}^{d-1} \binom{d-1}{i} l(H^i_{\mathfrak{m}}(R))$. $R$ is a generalized C.M. ring if and only if there exists a positive integer $k$ such that $\mathfrak{m}^k$ is an $R$-standard ideal. $R$ is called a Buchsbaum ring if $\mathfrak{m}$ is an $R$-standard ideal.

## 2. Main results

All results of §2 also hold for ideals of a regular local ring. For the simplicity of formulation we restrict our attention to the case of homogeneous ideals in a polynomial ring.

Recall that a homogeneous element $x \in \mathfrak{m}$ is called a superficial element of order 1 for $\mathfrak{m}$ if there exists a positive integer $p$ such that $(\mathfrak{m}^q : x) \cap \mathfrak{m}^p = \mathfrak{m}^{q-1}$ for $q >> 0$. For the properties of superficial elements see, e.g., [S].

**Lemma 2.1.** *If* $x \in \mathfrak{m}$ *is a superficial element of order* 1 *for* $\mathfrak{m}$, *then its image in* $R/(0: x)$ *is a superficial element of order* 1 *for* $\mathfrak{m}/(0: x)$.

Let $I \subset S = \mathbf{K}[x_1, \dots, x_n]$ be a homogeneous ideal. Set $R = S/I$.

**Lemma 2.2.** *Assume that* $I$ *is a generalized C.M. ideal and* $\dim(R) = d$. *Let* $\delta = \operatorname{depth}(R)$. *Then, for all* $1 \leq i \leq n$ *we have*

$$\beta_i(S/I) \leq \left(\binom{n-\delta+1}{i+1} - \binom{n-d+1}{i+1}\right) I(R) + \beta_i(S'/J),$$

*where* $J$ *is a* 0*-dimensional ideal of a polynomial ring* $S' = \mathbf{K}[y_1, \dots, y_{n-d}]$ *with* $e(J) = e(I)$.

*Proof.* Assume that $\delta = 0$. Let $x$ (resp. the image of $x$) be a homogeneous superficial element of order 1 for $\mathfrak{M}$ (resp. for $\mathfrak{m}$) (see [S], p. 7, for the existence of $x$). Consider the exact sequence

$$0 \to (0: x) \to R \to R/(0: x) \to 0.$$

Note that $l((0: x)) \leq l(H^0_{\mathfrak{m}}(R)) \leq I(R)$. Applying the functor $\operatorname{Tor}^S(\mathbf{K}, \cdot)$ to the above exact sequence we get

$$\begin{aligned}\beta_i(S/I) &= \dim \operatorname{Tor}_i^S(\mathbf{K}, R) \le \dim \operatorname{Tor}_i^S(\mathbf{K}, (0: x)) + \dim \operatorname{Tor}_i^S(\mathbf{K}, R/(0: x)) \\ &\le \binom{n}{i} I(R) + \dim \operatorname{Tor}_i^S(\mathbf{K}, R/(0: x)).\end{aligned}$$

By [HSV], Lemma 1,

$$\operatorname{Tor}_i^S(\mathbf{K}, R/(0: x)) \cong \operatorname{Tor}_i^{S_1}(\mathbf{K}, R/(0: x + xR)) = \beta_i(S_1/J_1),$$

where $S_1 := S/(x) J_1 = (I: x) + (x)/(x) \subset S_1$ is again a polynomial ring and $J_1$ is a homogeneous ideal. By Lemma 2.1, the image of $x$ is a superficial element of order 1 for $R/(0: x)$. Moreover, the image of $x$ is a non-zero divisor of $R/(0: x)$. Hence, $e(J_1) = e(R/(0: x)) = e(I)$. Repeating this process we get

$$\begin{aligned}\beta_i(S/I) &\le \left(\binom{n}{i} + \cdots + \binom{n-d+1}{i}\right) I(R) + \beta_i(S'/J) \\ &= \left(\binom{n+1}{i+1} - \binom{n-d+1}{i+1}\right) I(R) + \beta_i(S'/J),\end{aligned}$$

where $J$ is a 0-dimensional ideal of a polynomial ring $S' = \mathbf{K}[y_1, \dots, y_{n-d}]$ with $e(J) = e(I)$.

If $\delta > 0$, then in the first $\delta$ steps we have $\operatorname{Tor}_i^S(\mathbf{K}, 0: x) = 0$, and the result easily follows.

Let $\beta_i(e, n) = \beta_i(S/J(e, n))$ be Valla's bound for the $i$th Betti number of non-degenerate 0-dimensional ideals in $\mathbf{K}[x_1, \dots, x_n]$ with multiplicity $e$ ([V], Theorem 4). We set $\beta_i(e, n) = 1$ if $i = 0$ and $\beta_i(e, n) = 0$ if $i < 0$. We will extend Valla's results to any 0-dimensional ideal.

**Lemma 2.3.** *Let $I \subset \mathbf{K}[x_1, \dots, x_n]$ be a 0-dimensional ideal of multiplicity $e(I) = e$. Set $l = \dim_{\mathbf{K}}(I \backslash \mathfrak{M}^2) \otimes \mathbf{K}$. Then, for all $i = 1, \dots, n$, we have*

$$\beta_i(S/I) \le \sum_{j=0}^{l} \binom{l}{j} \beta_{i-j}(e, n-l).$$

*Proof.* We proceed by induction on $l$. For $l = 0$ it follows from (1.9). Assume $l > 0$; let $x \in I \backslash \mathfrak{M}^2$ be any linear form. By changing the coordinates we can assume that $x = x_n$. Let $I' = I/xS$, $S' = S/xS \cong \mathbf{K}[x_1, \dots, x_{n-1}]$. Then $\dim(I' \backslash \mathfrak{M}'^2) \otimes \mathbf{K} = l - 1$ and $e(I') = e(I)$. By [HSV], Lemma 1 and the induction hypothesis we have the following required result:

$$\begin{aligned}\beta_i(S/I) &\le \beta_{i-1}(S'/I') + \beta_i(S'/I') \\ &\le \sum_{j=0}^{l-1} \binom{l-1}{j} \beta_{i-1-j}(e, (n-1) - (l-1)) \\ &\quad + \sum_{j=0}^{l-1} \binom{l-1}{j} \beta_{i-j}(e, (n-1) - (l-1)) \\ &= \beta_{i-l}(e, n-l) + \sum_{j=1}^{l-1} \left(\binom{l-1}{j-1} + \binom{l-1}{j}\right) \beta_{i-j}(e, n-l) + \beta_i(e, n) \\ &= \sum_{j=0}^{l} \binom{l}{j} \beta_{i-j}(e, n-l).\end{aligned}$$

**Lemma 2.4.** *Let $I$ be a 0-dimensional ideal of $S$ of multiplicity $e$. Denote $n^* = \min(e-1, n)$, and define $t^*$, $r^*$ after the formulae in* (1.4) *for $e$ and $n^*$. Then, for all $i = 1, \dots, n$, we have*

$$\beta_i(S/I) \le \binom{t^*+i-2}{t^*-1}\binom{t^*+n-1}{t^*+i-1} + \sum_{p=i-1}^{n-1} \binom{p}{i-1} r^*_{\langle t^*\rangle (n-p)}.$$

*Proof.* Set $l = \dim_K(I \backslash \mathfrak{M}^2) \otimes \mathbf{K}$. If $n = l$, then $I = (x_1, \dots, x_n)$ and the above formula is trivially true. Assume $n - l \ge 1$. By Lemma 2.3 and (1.9) we have

$$\begin{aligned}
\beta_i(S/I) &\le \sum_{j=0}^{l} \binom{l}{j} \beta_{i-j}(e, n-l) = \sum_{j=0}^{\min(i-1,l)} \binom{l}{j} \beta_{i-j}(e, n-l) + \binom{l}{i} \\
&= \binom{l}{i} + \sum_{j=0}^{\min(i-1,l)} \binom{l}{j} \sum_{p=i-j-l}^{n-l-1} \binom{p}{i-j-1} e_p(e, n-l) \\
&= \binom{l}{i} + \sum_{p=i-\min(i-1,l)-1}^{n-l-1} e_p(e, n-l) \sum_{j=i-p-1}^{\min(i-1,l)} \binom{l}{j}\binom{p}{i-j-1}.
\end{aligned}$$

Since $\sum_{j=i-p-1}^{\min(i-1,l)} \binom{l}{j}\binom{p}{i-j-1} \le \binom{p+l}{i-1}$ and $e_p = 0$ for $p < 0$, by Lemma 1.8 we have

$$\begin{aligned}
\beta_i(S/I) &\le \binom{l}{i} + \sum_{p=0}^{n-l-1} \binom{p+l}{i-1} e_p(e, n-l) \\
&\le \binom{l}{i} + \sum_{p=0}^{n-l-1} \binom{p+l}{i-1} e_{n^*-(n-l)+p}(e, n^*) \\
&= \binom{l}{i} + \sum_{q=l}^{n-1} \binom{q}{i-1} e_{n^*-n+q}(e, n^*) \\
&= \binom{l}{i} + \sum_{q=\max(l,i-1)}^{n-1} \binom{q}{i-1} e_{n^*-n+q}(e, n^*).
\end{aligned}$$

Hence, using (1.5), we get

$$\begin{aligned}
\beta_i(S/I) \le{}& \binom{l}{i} + \sum_{q=\max(l,i-1)}^{n-1} \binom{q}{i-1}\binom{t^*-1+n^*-n+q}{t^*-1} \\
&+ \sum_{q=\max(l,i-1)}^{n-1} \binom{q}{i-1} r^*_{\langle t^*\rangle (n-q)} \\
\le{}& \binom{l}{i} + \sum_{q=\max(l,i-1)}^{n-1} \binom{q}{i-1}\binom{t^*-1+q}{t^*-1} + \sum_{q=i-1}^{n-1} \binom{q}{i-1} r^*_{\langle t^*\rangle (n-q)}.
\end{aligned}$$

If $l < i$, $\binom{l}{i} = 0$, and if $l \geq i$, we have

$$\sum_{q=i-1}^{l-1} \binom{q}{i-1}\binom{t^* - 1 + q}{t^* - 1} \geq \sum_{q=i-1}^{l-1} \binom{q}{i-1} = \binom{l}{i}.$$

So in both cases we obtain

$$\begin{aligned}\binom{l}{i} + \sum_{q=\max(l,\, i-1)}^{n-1} \binom{q}{i-1}\binom{t^* - 1 + q}{t^* - 1} &\leq \sum_{q=i-1}^{n-1} \binom{q}{i-1}\binom{t^* - 1 + q}{q} \\ &= \binom{t^* + i - 2}{t^* - 1}\binom{t^* + n - 1}{t^* + i - 1},\end{aligned}$$

which completes the proof of the lemma.

*Remark* 2.4.1. If $e - 1 \geq n$, then $n = n^*$ and we get exactly the same bound as Valla's bound for non-degenerate ideals ([V], Proposition 5(i)). Hence, we cannot improve Lemma 2.4 unless one involves $l$. But $l$ is not defined explicitly in our consideration.

*Remark* 2.4.2. Since $e_p(e, n^*) = \text{length}(S_p/J_p(e, n^*)) \leq e - (n^* - p)$ for all $0 \leq p \leq n^* - 1$, from the proof of Lemma 2.4 we get another estimation:

$$\beta_i(S/I) \leq \sum_{q=i-1}^{n-1} \binom{q}{i-1} e = \binom{n}{i} e.$$

For $n = n^*$ we get a little more:

$$\begin{aligned}\beta_i(S/I) &\leq \sum_{q=i-1}^{n-1} \binom{q}{i-1}(e - (n - q)) = \binom{n}{i}(e - n) + \sum_{q=i-1}^{n-1} q\binom{q}{i-1} \\ &= \binom{n}{i}(e - n) - \binom{n+1}{i+1} + n\binom{n}{i} = \binom{n}{i} e - \binom{n+1}{i+1}.\end{aligned}$$

It is interesting to compare this last result with the conjecture that $\beta_i(S/I) \geq \binom{n}{i}$.

**Lemma 2.5.** $r_{\langle t\rangle(n-q)} \leq \binom{t+q-1}{t}$ *for all* $0 \leq q \leq n$.

From Lemma 2.2 and Lemma 2.4 we get

**Theorem 2.6.** *Assume that* $I$ *is a generalized C.M. ideal of* $\text{ht}(I) = h$ *in* $\mathbf{K}[x_1, \ldots, x_n]$; $\delta = \text{depth}(R)$. *Let* $h^* = \min(e - 1, h)$ *and* $t^*$, $r^*$ *be defined for* $e$ *and* $h^*$. *Then for all* $1 \leq i \leq n$ *we have*

$$\begin{aligned}\beta_i(S/I) &\leq \binom{n + 1 - \delta}{i+1} - \binom{h+1}{i+1} I(R) \\ &\quad + \binom{t^* + i - 2}{t^* - 1}\binom{t^* + h - 1}{t^* + i - 1} + \sum_{p=i-1}^{h-1} \binom{p}{i-1} r^*_{\langle t^*\rangle(h-p)} \\ &\leq \left(\binom{n + 1 - \delta}{i+1} - \binom{h+1}{i+1}\right) I(R) + \binom{t^* + i - 1}{t^*}\binom{t^* + h}{t^* + i}.\end{aligned}$$

*Proof*. The first inequality follows immediately from Lemma 2.2 and Lemma 2.4. Further, by Lemma 2.5 we have

$$\begin{aligned}\sum_{p=i-1}^{h-1}\binom{p}{i-1}\delta^*_{\langle t^*\rangle(h-p)} &\leq \sum_{p=i-1}^{h-1}\binom{p}{i-1}\binom{t^*+p+h^*-h-1}{t^*}\\ &\leq \sum_{p=i-1}^{h-1}\binom{p}{i-1}\binom{t^*+p-1}{t^*}\\ &= \sum_{p=i-1}^{h-1}\binom{p}{i-1}\left(\binom{t^*+p}{p}-\binom{t^*+p-1}{p}\right)\\ &= \binom{t^*+i-1}{t^*}\binom{t^*+h}{t^*+i}-\binom{t^*+i-2}{t^*-1}\binom{t^*+h-1}{t^*+i-1}.\end{aligned}$$

From that we get the second inequality.

*Remark* 2.7. Other bounds are (by Remarks 2.4.1 and 2.4.2)

$$\beta_i(S/I)\leq\left(\binom{n+1-\delta}{i+1}-\binom{h+1}{i+1}\right)I(R)+\binom{h}{i}e,$$

or for $e\geq n-1$

$$\beta_i(S/I)\leq\left(\binom{n+1-\delta}{i+1}-\binom{h+1}{i+1}\right)I(R)+\binom{h}{i}e-\binom{h+1}{i+1}.$$

**Example.** For $I=(x_1,\ldots,x_{n-1})\cap(x_1,\ldots,x_n)^2$ we have $I(R)=n-1$, $t^*=1$, $r^*=0$. Hence, the first bound in Theorem 2.6 equals the bound in Remark 2.7 and equals $\binom{n}{i}(n-1)+\binom{n-1}{i}=\binom{n}{i}((n-1)+(i+1)(n-i)/n)$. Using [EK] we get

$$\beta_i(S/I)=\binom{n}{i}(n-1)+\binom{n-1}{i+1}=\binom{n}{i}((n-1)+(n-i)(n-i-1)/((i+1)n)).$$

This shows that in this example the bounds given in Theorem 2.6 are not far of being sharp.

In order to get bounds for the Betti numbers in terms of $e$, $n$ and $k$ where $k$ is defined in (1.10) one can combine Theorem 2.6 with the following bounds on $I(R)$ given by the first author in [H], where one can also find better bounds for $\beta_1(S/I)$. Note that there is already no upper bound for the number of generators of $I$ which does not involve $k$. Following [EVR], §4, we set $\nu(e,h)=\binom{h+t-1}{t}-r+r^{\langle t\rangle}$ where $t$ and $r$ are defined in (14).

**Lemma 2.8.** *Let $I$ be an ideal of $S$ such that $\mathfrak{m}^k$ is an $R$-standard ideal ($k>0$). Then*

(i) $I(R)\leq(n-1-n'+\nu(k^{d-1}e,n'))\binom{n+k-1}{k-1}$, *where*

$$n'=\min(n-1,k^{d-1}e-1).$$

(ii) $I(R)\leq((2n-3)/(n-1)+ek^{d-1}(n-2)^2/(n-1))\binom{n+k-1}{k-1}$.

(iii) *If* depth$(R)>0$, *then* $I(R)\leq(d-1)(k^de-1)$.

(iv) *If $I$ is a Buchsbaum ideal, then* $I(R)\leq h-h'+\nu(e,h')$, *where* $h'=\min(h,e-1)$.

## 3. Applications

As applications of our results we will give upper bounds for the Betti numbers of the homogeneous ideal of some projective schemes.

**Example 3.1.** Let $V$ be an arithmetically Buchsbaum projective subscheme of codimension 2 in $\mathbf{P}^n$. Then by [H], Corollary 4.1, $I(R) \leq (1+8e)^{1/2}$ and $t^* \leq (-1+(1+8e)^{1/2})/2$. Hence by Theorem 2.6 we get

$$\beta_i(S/I_V) \leq \left(\binom{n+1}{i+1} - \binom{3}{i+1}\right)(1+8e)^{1/2} + \binom{t^*+i-1}{t^*}\binom{t^*+2}{t^*+i}.$$

Thus:

(1) $\beta_1(S/I_V) \leq (\binom{n+1}{2}-3)(1+8e)^{1/2}+(3+(1+8e)^{1/2})/2 = \binom{n+1}{2}(1+8e)^{1/2}+3/2-5(1+8e)^{1/2}/2$.

(2) $\beta_2(S/I_V) \leq (\binom{n+1}{3}-1)(1+8e)^{1/2}+(1+(1+8e)^{1/2})/2 = \binom{n+1}{3}(1+8e)^{1/2}+1/2-((1+8e)^{1/2})/2$.

(3) For all $n \geq i \geq 3$, $\beta_i(S/I_V) \leq \binom{n+1}{i+1}(1+8e)^{1/2}$.

**Example 3.2.** Let $V$ be an arithmetically Buchsbaum projective subscheme of codimension $h$ in $\mathbf{P}^n$. Then, we have the following bounds on the Betti numbers:

$$\beta_i(S/I_V) \leq \left(\binom{n+2-\delta}{i+1} - \binom{h+1}{i+1}\right)(d-1)(e-1) + \binom{h}{i}e - \binom{h+1}{1+i}.$$

In particular, if $d = 2$ (i.e., for curves)

$$\beta_i(S/I_V) \leq \binom{n}{i}(e-1) + \binom{n-1}{i}e - \binom{n}{1+i}.$$

**Example 3.3.** Let $C$ be a non-degenerate, reduced and irreducible curve in $\mathbf{P}^n$ over an algebraically closed field $\mathbf{K}$. From Theorem 2.6 and Remark 2.7 we get

$$\beta_i(S/I_C) \leq \binom{n}{i}I(R) + \binom{n-1}{i}e - \binom{n}{i+1}.$$

Using Lemma 2.8 (iii) we obtain

$$\beta_i(S/I_C) \leq \binom{n}{i}(k^2e-1) + \binom{n-1}{i}e - \binom{n}{i+1}.$$

By [GLP], one can choose $k \leq \max(1, e-n)$; hence if $e \geq n+1$ we have

$$\beta_i(S/I_C) \leq \binom{n}{i}((e-n)^2e-1) + \binom{n-1}{i}e - \binom{n}{i+1},$$

which only depends on $e$.

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