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A RESIDUE MAP AND ITS APPLICATIONS TO SOME ONE-DIMENSIONAL RINGS

I-CHIAU HUANG

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ABSTRACT. A residue map is used to study canonical modules of the ring $k[[X^{t_1}, \dots, X^{t_n}]]$. A simple proof of a well-known numerical criterion for $k[[X^{t_1}, \dots, X^{t_n}]]$ to be Gorenstein is given.

Let k be a field and R be the one-dimensional local ring $k[[X^{t_1}, \dots, X^{t_n}]]$, where $(t_1, \dots, t_n) = 1$. Denote by $\mathfrak{m}$ the maximal ideal of R. It is well known that injective hulls of an R-module are isomorphic to each other. However, finding isomorphisms between them is a very subtle question. For example, the R-module $\mathrm{Hom}_k^c(R, k)$ consisting of all R-linear maps $u: R \to k$, which are continuous for the $\mathfrak{m}$-adic topology of R and the discrete topology of k (i.e., $u(\mathfrak{m}^n) = 0$ for some n), is an injective hull of $R/\mathfrak{m}$ [3, Proposition 3.4]. Another injective hull of $R/\mathfrak{m}$ is given by local cohomology: Let $k[[X_1, \dots, X_n]] \to R$ be the canonical map sending X_i to X^{t_i}. $\mathrm{Hom}_{k[[X_1,\dots,X_n]]}(R, H^n_{X_1,\dots,X_n}(k[[X_1, \dots, X_n]]))$ is also an injective hull of $R/\mathfrak{m}$ [3, Propositions 3.4 and 3.8]. An isomorphism

$$\mathrm{Hom}_{k[[X_1,\dots,X_n]]}(R, H^n_{X_1,\dots,X_n}(k[[X_1, \dots, X_n]])) \simeq \mathrm{Hom}_k^c(R, k)$$

can be described using a residue map (cf. [3]).

A similar phenomenon occurs in canonical modules. Denote by $(-)^\wedge$ the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_k^c(R, k))$. A finitely generated R-module K is called a canonical module if there is a functorial isomorphism

$$(\mathrm{Ext}^{1-i}(M, K))^\wedge \simeq H^i_{\mathfrak{m}}(M)$$

for finitely generated R-module M $(i = 0, 1)$. It is easy to see that any canonical module, if it exists, is isomorphic to $(H^1_{\mathfrak{m}}(R))^\wedge$. But in general isomorphisms between canonical modules are not easy to describe. This note gives an explicit description of a canonical module K as a submodule of $k[[X]]$ and describes explicitly an isomorphism from K to $(H^1_{\mathfrak{m}}(R))^\wedge$ using a residue map. The main purpose of this note is to bring out some concrete aspects of Grothendieck's duality theory into the open, so that the structure of R is better understood.

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This note uses freely basic properties of Matlis duality, which is referred to [6].

Let α be the conductor of R (i.e., the smallest number such that $X^i \in R$ for all $i \geq \alpha$). The residue map we need in this note is the k-linear map

$$\text{res}\colon k((X)) \to k$$

defined by $\text{res}(\sum a_i X^i) = a_{\alpha-1}$ for $\sum a_i X^i$ in $k((X))$. It induces an R-isomorphism from $k((X))$ to its dual.

Lemma 1. *The R-linear map*

$$\Phi\colon k((X)) \to k((X))^\wedge,$$

defined by $\Phi(f)(g)(r) = \text{res}(fgr)$ for $f, g \in k((X))$ and $r \in R$, is an isomorphism.

Proof. Let $f = a_{-m}X^{-m} + \cdots + a_0 + a_1 X + \cdots$ be an element in $k((X))$. Since $\Phi(f)(X^{\alpha+m-1})(1) = a_{-m}$, $\Phi(f) = 0$ implies $f = 0$. Given $\phi \in k((X))^\wedge$ and $X^j \in R$,

$$\phi(X^i)(X^j) = (X^j\phi(X^i))(1) = \phi(X^{i+j})(1).$$

Hence $\phi(X^l)$ vanishes for l sufficiently large. Define

$$f := \sum_{i\in\mathbb{Z}} \phi(X^i)(1)X^{\alpha-1-i}.$$

Then $\Phi(f)(X^l)(1) = \phi(X^l)(1)$. Since ϕ is determined by $\phi(X^l)(1)$, it follows that $\Phi(f) = \phi$. □

Proposition 2. *The map $\Psi\colon k((X)) \to \text{Hom}_k^c(R, k)$ sending $f \in k((X))$ to the map $r \mapsto \text{res}(fr)$ is R-linear and surjective.*

Proof. The proposition can be verified easily and directly. It also follows from the fact that Ψ is the composition

$$k((X)) \xrightarrow{\Phi} k((X))^\wedge \to \text{Hom}_k^c(R, k),$$

where the second map $k((X))^\wedge \to \text{Hom}_k^c(R, k)$ is the dual of the inclusion map $R \to k((X))$. □

We remark that

$$k((X)) \xrightarrow{\Psi} \text{Hom}_k^c(R, k)$$

is actually a dualizing complex of R. The proof is left to the reader who is familiar with dualizing complexes. The kernel K of the map Ψ has a simple description: Let S be the semigroup generated by $t_1, \ldots, t_n$. Denote by $S^\wedge$ the subset of $\mathbb{Z}$ consisting of those elements i such that $\alpha - 1 - i \notin S$. Then

$$K = \left\{\sum a_i X^i | a_i \in k \text{ and } i \in S^\wedge\right\}.$$

Since $S^\wedge$ is a subset of $\mathbb{N}$, K is an R-submodule of $k[[X]]$.

Recall that R is Gorenstein if it has finite injective dimension. A necessary and sufficient condition for R to be Gorenstein is that it has an injective resolution

$$0 \to R \xrightarrow{\text{inclusion}} k((X)) \longrightarrow \text{Hom}_k^c(R, k) \longrightarrow 0$$

for some R-linear map $k((X)) \to \text{Hom}_k^c(R, k)$ [6, Theorem 18.8]. As a consequence of the proposition and the fact that $k((X))$ and $\text{Hom}_k^c(R, k)$ are injective R-modules, we get a simple proof of the following theorem due to Kunz [5].

Corollary 3. *If S is a symmetric (i.e., $S = S^\wedge$), R is Gorenstein.*

Theorem 4. *K is a canonical module of R.*[1]

Proof. K is finitely generated because it is generated by X^i, $i \in S^\wedge$ and $i < 2\alpha$. Let M be a finitely generated R-module. $(\mathrm{Ext}^{1-i}(M, K))^\wedge$ is the ith homology of the following complex induced by Ψ:

$$M^{\wedge\wedge} \xrightarrow{d} \mathrm{Hom}_R(M, k((X)))^\wedge.$$

By Matlis duality, the canonical map $M \to M^{\wedge\wedge}$ is an isomorphism. We claim that the map

$$\Phi_M : M_{X^\alpha} \to \mathrm{Hom}_R(M, k((X)))^\wedge$$

defined by $\Phi_M(\frac{m}{X^{l\alpha}})(\phi)(r) = \mathrm{res}\frac{r\phi(m)}{X^{l\alpha}}$ for $\phi \in \mathrm{Hom}_R(M, k((X)))$ and $r \in R$ is an R-isomorphism. Since the functors $(-)_{X^\alpha}$ and $\mathrm{Hom}_R(-, k((X)))^\wedge$ are additive and exact, and M is finitely generated, to prove the claim we may assume that $M = R$. In such case $R_{X^\alpha} = k((X))$, so the claim follows from Lemma 1. The theorem follows form the observation that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\text{can.}} & M_{X^\alpha} \\ {\scriptstyle\text{can.}}\downarrow & & \downarrow{\scriptstyle\Phi_M} \\ M^{\wedge\wedge} & \xrightarrow{d} & \mathrm{Hom}_R(M, k((X)))^\wedge \end{array}$$

is commutative. □

We have a simple proof of the following result which is contained in [1] and [4].

Corollary 5. *Some ideals of R are canonical modules.*

Proof. The map $K \to k((X))$, multiplied by X^α, is R-linear and injective with image in R. □

The converse of Corollary 3, also due to Kunz [5], has a simple proof.

Theorem 6. *If R is Gorenstein, S is symmetric.*

Proof. R has an injective resolution

$$0 \to R \xrightarrow{\text{inclusion}} k((X)) \longrightarrow \mathrm{Hom}_k^c(R, k) \longrightarrow 0$$

with some unknown map $k((X)) \to \mathrm{Hom}_k^c(R, k)$. Taking dual, then applying Matlis duality and Lemma 1, we get an exact sequence:

$$0 \longrightarrow R \longrightarrow k((X)) \xrightarrow{\Psi} \mathrm{Hom}_k^c(R, k) \longrightarrow 0.$$

This implies $K \simeq R$, although the map $R \to k((X))$ above is still unknown. Since $1 \in K$, K is generated by a power series f whose constant coefficient is 1 and there exists

$$g = 1 + \sum_{i>0,\, i\in S} a_i X^i$$

[1]Professor Herzog informed me that this description of canonical module is implicitly in [2].

in R, such that $gf = 1$. S is a semigroup, hence

$$f = 1 - \left(\sum a_i X^i\right) + \left(\sum a_i X^i\right)^2 - \left(\sum a_i X^i\right)^3 + \cdots \in R.$$

Therefore $K = R$ (as subsets of $k[[X]]$). Equivalently $S = S^\wedge$. □

$H^1_{\mathfrak{m}}(R)$ is isomorphic to the cokernel of the map

$$R \xrightarrow{\text{inclusion}} R_{X^\alpha}.$$

Since $R_{X^\alpha} = k((X))$, $H^1_{\mathfrak{m}}(R) \simeq k((X))/R$. The image of K of the map

$$\Phi: k((X)) \to k((X))^\wedge$$

consists of all $\phi \in k((X))^\wedge$ with $\phi(R) = 0$. Therefore we get an isomorphism

$$K \xrightarrow{\sim} (k((X))/R)^\wedge.$$

Example 7. $R = k[[X^7, X^9, X^{10}]]$.

$S = \{0, 7, 9, 10, 14, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, \dots\}$.

The conductor $\alpha = 23$.

$S^\wedge = \{0, 7, 9, 10, 11, 14, 16, 17, 18, 19, 20, 21, 23, \dots\}$.

$X^7 K = RX^7 + RX^{18}$ is a canonical module of R.

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Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan, Republic of China

E-mail address: maichiau@ccvax.sinica.edu.tw