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A RANDOM BANACH-STEINHAUS THEOREM

M. V. VELASCO AND A. R. VILLENA

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ABSTRACT. In an earlier paper, we began a study of linear random operators which have a certain probability of behaving as continuous operators. In this paper we study the pointwise limit in probability of a sequence of such operators, extending the Banach-Steinhaus theorem in a stochastical sense.

I. INTRODUCTION

Banach space-valued random variables and random operators on Banach spaces are topics of lively interest due to their intimate connections with random equations theory ([1], [3]), with the stochastic integration operators being the main examples of such random operators.

Stochastic continuity is a very important property of random operators which can be treated by such traditional methods as the closed graph theorem or the Banach-Steinhaus theorem. Surprisingly, random operators that are not continuous appear naturally in a recent study of the stochastic continuity of linear random operators on Banach algebras ([4], [6]). Therefore, we can't use classical theorems for them. However, we proved that not stochastically continuous linear random operators on Banach algebras behave as continuous ones in some sense which can be quantified with precision ([4]). In this way probably continuous random operators arise as those linear random operators which have a certain probability nonzero of behaving as continuous operators. This topic was considered in [5], proving random versions of classical theorems, such as the closed graph theorem or the open mapping theorem, for such operators.

In this paper we study the pointwise limit in probability of a sequence of probably continuous random operators by randomizing the classical Banach-Steinhaus theorem. This randomization becomes a fundamental principle in the treatment of the automatic continuity of linear random operators which can be found in [6].

II. STOCHASTIC VERSION OF THE BANACH-STEINHAUS THEOREM

Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a probability space which is considered fixed throughout this paper. Let $(Y, \|\cdot\|)$ be a Banach space. The linear space of all Y-valued *Bochner random variables* on $(\Omega, \mathscr{A}, \mathscr{P})$ is said to be a *randomization* of

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Y, and we denote it by $\mathscr{R}_\Omega(Y)$ ($\mathscr{R}(Y)$ if there is confusion). We consider every randomization to be endowed with the topology of the convergence in probability, which is metrizable (see [2], Section 2), and provide $\mathscr{R}(Y)$ with F-space structure (where elements that coincide almost surely are equivalent). Therefore, a sequence in $\mathscr{R}(Y)$, $\{\mathbf{y}_n\}$, converges to an element $\mathbf{y}$ in $\mathscr{R}(Y)$ if for each $\tau > 0$

$$\lim_{n\to\infty} \mathscr{P}[\|\mathbf{y}_n - \mathbf{y}\| > \tau] = 0.$$

An operator from a Banach space $(X, \|\cdot\|)$ into a randomization $\mathscr{R}(Y)$ is called a *random operator* from X to Y. We are working only with *linear random operators*, i.e., random operators, T, with the following property:

$$\mathscr{P}[T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)] = 1$$

(for all x, y in X and α, β constants).

We say that T is *stochastically continuous* at x_0 in X if

$$\lim_{n\to\infty} \|x_n - x_0\| = 0$$

implies that for each $\tau > 0$

$$\lim_{n\to\infty} \mathscr{P}[\|T(x_n) - T(x_0)\| > \tau] = 0.$$

It is said that T is *stochastically continuous* if T is stochastically continuous at every x_0 in X.

It is easy to check that:

A linear random operator, T, is stochastically continuous if and only if for every $0 < \varepsilon < 1$ there exists a constant, M_ε, such that $\mathscr{P}[\|T(x)\| \le M_\varepsilon\|x\|] \ge \varepsilon$, for all x in X.

From now on, let $\{T_i\}_{i\in I}$ be a family of linear random operators from the Banach space X to the Banach space Y.

We can phrase the *equicontinuity* property of the family $\{T_i\}_{i\in I}$, with regard to both the X-uniformity and the $\mathscr{R}(Y)$-uniformity, as follows.

For each $0 < \varepsilon < 1$ there exists a positive constant, M_ε, such that

$$\mathscr{P}[\|T_i(x)\| \le M_\varepsilon\|x\|] \ge \varepsilon, \quad \forall x \in X, \quad \textit{for all } i \textit{ in } I.$$

We say that such a $\{T_i\}_{i\in I}$ is *stochastically equicontinuous.*

In the same way, we say that $\{T_i\}_{i\in I}$ is *stochastically pointwise bounded* if all x in X satisfy the following condition. *If $0 < \varepsilon < 1$ there exists a constant, $M_{x,\varepsilon}$, such that*

$$\mathscr{P}[\|T_i(x)\| \le M_{x,\varepsilon}] \ge \varepsilon, \quad \forall i \in I.$$

The classical Banach-Steinhaus theorem ([7], Theorem 3.3.6) can be rephrased now as follows.

Stochastic version of the Banach-Steinhaus theorem. *A stochastically pointwise bounded family of stochastically continuous linear random operators is stochastically equicontinuous.*

III. Random Banach-Steinhaus theorem

Definition 1. We say that a linear random operator, T, is *probably continuous* if there exists $0 < \delta < 1$, with the following property. For every $0 < \delta' < \delta$, there exists $M_{\delta'} > 0$ such that

$$P[\|T(x)\| \le M_{\delta'}\|x\|] \ge \delta', \quad \forall x \in X.$$

For a probably continuous linear random operator, T, we define

$$\alpha(T) = \sup\{\delta\colon \text{ if } \ 0<\delta'<\delta,\ \exists M_{\delta'}>0 \ \text{ with } \ \mathscr{P}[\|T(x)\| \le M_{\delta'}\|x\|] \ge \delta',\ \forall x\in X\}.$$

For a nonprobably continuous linear random operator, T, we write $\alpha(T)=0$.

Given a linear random operator, T, from X to Y and a measurable subset, Ω_0, with $\mathscr{P}[\Omega_0]>0$, we can consider Ω_0 as a new probability space with the inherited structure from Ω, and the operator $T/\Omega_0\colon X\to \mathscr{R}_{\Omega_0}(Y)$, defined by $T/\Omega_0(x)=T(x)/\Omega_0$, is called a *conditional operator* of T. It is easy to check that *every linear random operator having a stochastically continuous conditional operator is probably continuous.* Our next theorem follows from Corollary 8 of [5].

Theorem 2. *If T is probably continuous, then it has a stochastically continuous conditional operator, T/Ω_0, with $\mathscr{P}[\Omega_0]=\alpha(T)$ and it is not possible to find a stochastically continuous conditional operator, T/Ω_0', with $\mathscr{P}[\Omega_0']>\alpha(T)$.*

In line with this same pattern of the probable continuity concept we have the following definition.

Definition 3. We say that $\{T_i\}_{i\in I}$ is *probably equicontinuous* if we can find $0<\delta<1$ such that for each $0<\delta'<\delta$ there exists $M_{\delta'}>0$ satisfying

$$\mathscr{P}[\|T_i(x)\|\le M_{\delta'}\|x\|]\ge\delta', \quad \forall x\in X, \quad \text{for all } i \text{ in } I.$$

This is the moment to wonder what is meant by the family $\{T_i\}_{i\in I}$ being *probably pointwise bounded*, and there are two "reasonable" answers.

(1) There exists a measurable set Ω_0 with $\mathscr{P}[\Omega_0]\ge\delta$ such that the correspondent family of conditional operators $\{T_i/\Omega_0\}_{i\in I}$ is stochastically pointwise bounded. (This is the uniform sense.)

(2) There exists $0<\delta<1$ such that for all x in X there exists a constant, M_x, satisfying

$$\mathscr{P}[\|T_i(x)\|\le M_x]\ge\delta, \quad \forall i\in I.$$

(This is the weak sense.)

Obviously the second concept is weaker than the first, and unfortunately we cannot obtain a "uniform randomization principle" (see [5], Theorem 4) which permits us to put in equivalence both concepts, as the following example shows.

Example 4. Let $(\Omega,\mathscr{A},\mathscr{P})=([0,1],\mathscr{B},\lambda)$ where $\mathscr{B}$ is the Borel σ-algebra of $[0,1]$ and λ is the Lebesgue measure. We consider the family of operators on $\mathbb{R}$, $\{T_n\}_{n\in\mathbb{N}}$, defined by

$$T_n(x)=x\mathbf{y}_n, \quad \text{for all } x \text{ in } \mathbb{R} \text{ and every } n \text{ in } \mathbb{N},$$

where $\mathbf{y}_n$ are ordinary random variables defined as follows:

$$\mathbf{y}_{2n-1}(\omega)=\begin{cases}0 & \text{if } \omega\in[0,\frac13],\\ n & \text{if } \omega\in]\frac13,1],\end{cases}$$

$$\mathbf{y}_{2n}(\omega)=\begin{cases}n & \text{if } \omega\in[0,\frac23],\\ 0 & \text{if } \omega\in]\frac23,1].\end{cases}$$

This family is stochastically pointwise bounded in the weak sense because, given some positive M,

$$\mathscr{P}[\|T_n(x)\| \leq M] \geq \tfrac{1}{3}, \quad \forall x \in X,$$

but it is impossible to find a measurable subset Ω_0 with $\mathscr{P}[\Omega_0] > 0$ with $\{T_n/\Omega_0\}_{n\in\mathbf{N}}$ being stochastically pointwise bounded.

Lemma 5. *If T is a probably continuous linear random operator from X to Y and $\{x_n\}_{n\in\mathbf{N}}$ is a sequence of X which converges to x in X, then, for all $\tau > 0$,*

$$|\underline{\lim}\mathscr{P}[\|T(x_n)\| \leq \tau] - \mathscr{P}[\|T(x)\| \leq \tau]| \leq 1 - \alpha(T).$$

Proof. We assume that $\tau > 0$ exists, such that

$$\underline{\lim}\mathscr{P}[\|T(x_n)\| \leq \tau] > \mathscr{P}[\|T(x)\| \leq \tau] + 1 - \alpha(T),$$

for some sequence $\{x_n\}_{n\in\mathbf{N}}$ which converges to x. Then, for a sufficiently small real γ and every sufficiently large natural number n,

$$\mathscr{P}[\|T(x_n)\| \leq \tau] + \mathscr{P}[\|T(x)\| > \tau] > 2 - \alpha(T) + 2\gamma. \tag{1}$$

Since

$$\lim_{\alpha\to 0}\mathscr{P}[\|T(x)\| \geq \tau + \alpha] = \mathscr{P}[\|T(x)\| > \tau],$$

there exists $\alpha_0 > 0$ such that

$$\mathscr{P}[\|T(x)\| \geq \tau + \alpha_0] \geq \mathscr{P}[\|T(x)\| > \tau] - \gamma,$$

and by (1)

$$\mathscr{P}[\|T(x_n)\| \leq \tau] + \mathscr{P}[\|T(x)\| \geq \tau + \alpha_0] > 2 - \alpha(T) + \gamma.$$

Therefore

$$\begin{aligned}\mathscr{P}[\|T(x_n) - T(x)\| \geq \alpha_0] &\geq \mathscr{P}[|\,\|T(x_n)\| - \|T(x)\|\,| \geq \alpha_0] \\ &\geq \mathscr{P}[\|T(x_n)\| \leq \tau,\ \|T(x)\| \geq \tau + \alpha_0] \\ &\geq \mathscr{P}[\|T(x_n)\| \leq \tau] + \mathscr{P}[\|T(x)\| \geq \tau + \alpha_0] - 1 > 1 - \alpha(T) + \gamma.\end{aligned}$$

Thus

$$\mathscr{P}[\|T(x_n) - T(x)\| < \alpha_0] < \alpha(T) - \gamma,$$

so

$$\underline{\lim}\mathscr{P}[\|T(x_n - x)\| < \alpha_0] \leq \alpha(T) - \gamma < \alpha(T),$$

in contradiction with Corollary 8(iii) of [4] by which

$$\alpha(T) \geq \underline{\lim}_{x\to 0}\mathscr{P}[\|T(x)\| < \varepsilon], \quad \text{for all } \varepsilon > 0.$$

This proves

$$\underline{\lim}\mathscr{P}[\|T(x_n)\| \leq \tau] - \mathscr{P}[\|T(x)\| \leq \tau] \leq 1 - \alpha(T).$$

An analogous argument shows

$$\mathscr{P}[\|T(x)\| \leq \tau] - \underline{\lim}\mathscr{P}[\|T(x_n)\| \leq \tau] \leq 1 - \alpha(T). \quad \square$$

The stochastic version of the Banach-Steinhaus theorem is a particular case of the following theorem.

Random Banach-Steinhaus Theorem 6. *Let* $\{T_i\}_{i\in I}$ *be a family of probably continuous linear random operators. Let* $\delta > 0$ *be such that for each* x *in* X, *there exists* M_x *such that*

$$\mathscr{P}[\|T_i(x)\| \le M_x] \ge \delta, \quad \forall i \in I.$$

Then there exists $M > 0$ *such that for each* x *in* X

$$\mathscr{P}[\|T_i(x)\| \le M\|x\|] \ge (2\delta - 1) - (1 - \alpha(T_i)), \quad \forall i \in I.$$

Proof. We define

$$C_n = \{x \in X \colon \mathscr{P}[\|T_i(x)\| \le n] \ge \delta, \quad \forall i \in I\}.$$

Since $X = \bigcup_{n=1}^{\infty} \overline{C_n}$, by the Baire theorem, we can find some $\overline{C_m}$ which contain a ball, $B(x_0, 2r)$. Therefore, for all x such that $\|x\| = 1$,

$$x_0 + rx \in \overline{C_m},$$

and we can obtain a sequence $\{x_k\}$ in C_m which converges to $x_0 + rx$. Applying the last lemma,

$$\mathscr{P}[\|T_i(x_0 + rx)\| \le m] \ge \underline{\lim}\mathscr{P}[\|T_i(x_k)\| \le m] - (1 - \alpha(T_i)) \ge \delta - (1 - \alpha(T_i)),$$

and so

$$\mathscr{P}[\|T_i(rx)\| \le m + \|T_i(x_0)\|] \ge \delta - (1 - \alpha(T_i)).$$

Observing that

$$\begin{aligned}\mathscr{P}[\|T_i(rx)\| \le 2m] &\ge \mathscr{P}[\|T_i(rx)\| \le m + \|T_i(x_0)\|, \|T_i(x_0)\| \le m] \\ &\ge \delta - (1 - \alpha(T_i)) + \delta - 1\end{aligned}$$

and taking $M = \frac{2m}{r}$ we conclude that

$$\mathscr{P}[\|T_i(x)\| \le M\|x\|] \ge (2\delta - 1) - (1 - \alpha(T_i)), \quad \forall x \in X. \quad \square$$

Corollary 7. *Let* $\{T_i\}_{i\in I}$ *be a stochastically pointwise bounded family of probably continuous linear random operators. Then, for every* $0 < \varepsilon < 1$, *there exists a constant,* M_ε, *such that*

$$\mathscr{P}[\|T_i(x)\| \le M_\varepsilon\|x\|] \ge \varepsilon - (1 - \alpha(T_i)), \quad \forall i \in I, \forall x \in X.$$

Proof. Given $0 < \varepsilon < 1$, we consider $\delta = \frac{\varepsilon+1}{2}$, and a straightforward application of the last theorem proves the result. $\square$

Supposing a stronger pointwise boundedness in the hypothesis of the random Banach-Steinhaus theorem (i.e., assuming that $\{T_i\}_{i\in I}$ is probably pointwise bounded in the uniform sense), we can retrieve the lost part of δ in the thesis, as the following result shows.

Corollary 8. *Let* $\{T_i\}_{i\in I}$ *be a family of probably continuous linear random operators. If there exists a measurable subset* Ω_0, *with* $\mathscr{P}[\Omega_0] \ge \delta$, *such that* $\{T_i/\Omega_0\}_{i\in I}$ *is stochastically pointwise bounded, then, for every* $0 < \delta' < \delta$, *there exists* $M_{\delta'} > 0$ *such that, for each* x *in* X,

$$\mathscr{P}[\|T_i(x)\| \le M_{\delta'}\|x\|] \ge \delta' - (1 - \alpha(T_i)), \quad \forall i \in I.$$

Proof. Let us denote the conditional probability relative to Ω_0 by $\mathscr{P}'$. Applying the last corollary to $\{T_i/\Omega_0\}_{i\in I}$ it follows that, for every $0 < \varepsilon < 1$, there exists a constant, M_ε, such that

$$\mathscr{P}'[\|T_i/\Omega_0(x)\| \le M_\varepsilon\|x\|] \ge \varepsilon - (1 - \alpha(T_i/\Omega_0)), \quad \forall x \in X, \forall i \in I.$$

Then

(1) $\mathscr{P}[\|T_i(x)\| \leq M_\varepsilon\|x\|] \geq \mathscr{P}(\Omega_0)(\varepsilon - (1 - \alpha(T_i/\Omega_0)))$, $\forall x \in X$, $\forall i \in I$.

On the other hand, by Theorem 2, for every i in I there exists a measurable set Ω_i such that T/Ω_i is stochastically continuous and $\mathscr{P}(\Omega_i) = \alpha(T_i)$. Also,

$$\alpha(T_i/\Omega_0) \geq \mathscr{P}'[\Omega_i \cap \Omega_0] \geq \frac{\alpha(T_i) + \mathscr{P}[\Omega_0] - 1}{\mathscr{P}[\Omega_0]},$$

that is

$$\mathscr{P}[\Omega_0](1 - \alpha(T_i/\Omega_0)) \leq 1 - \alpha(T_i), \quad \forall i \in I.$$

Now, by (1) we have established that, for every $0 < \varepsilon < 1$, there exists a constant M_ε such that

$$\mathscr{P}[\|T_i(x)\| \leq M_\varepsilon\|x\|] \geq \delta\varepsilon - (1 - \alpha(T_i)). \quad \forall x \in X, \forall i \in I.$$

Thus, given $0 < \delta' < \delta$, we take $\varepsilon = \frac{\delta'}{\delta}$ in the last inequality, and the result is proved. □

Definition 9. Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of linear random operators from X to Y. We say that a linear random operator, T, from X to Y, is the *stochastic pointwise limit* of T if, for each x in X, $\{T_n\}_{n\in\mathbb{N}}$ converges to $T(x)$ in probability.

Corollary 10. *The stochastic pointwise limit, T, of a sequence $\{T_n\}_{n\in\mathbb{N}}$ of probably continuous linear random operators satisfies the inequality*

$$\alpha(T) \geq \underline{\lim}\,\alpha(T_n).$$

Proof. If $\alpha(T) < \underline{\lim}\,\alpha(T_n)$, we can find a real β such that

$$\alpha(T) < \beta < \underline{\lim}\,\alpha(T_n),$$

so, taking a subsequence if it is necessary, it is not restrictive to assume that

$$\alpha(T_n) > \beta, \quad \forall n \in \mathbb{N}.$$

Moreover, $\{T_n\}_{n\in\mathbb{N}}$ is stochastically pointwise bounded, so, by Corollary 7, if $0 < \varepsilon < 1$ there exists a constant, M_ε, such that

$$\mathscr{P}[\|T_n(x)\| \leq M_\varepsilon\|x\|] \geq \varepsilon - (1 - \alpha(T_n)) \geq \varepsilon - (1 - \beta), \quad \forall x \in X, \forall n \in \mathbb{N}.$$

Therefore, if $\tau > 0$,

$$\begin{aligned}\mathscr{P}[\|T(x)\| \leq M_\varepsilon\|x\| + \tau] &\geq \mathscr{P}[|\,\|(T_n - T)(x)\| - \|T_n(x)\|\,| \leq M_\varepsilon\|x\| + \tau] \\ &\geq \mathscr{P}[\|(T_n - T)(x)\| < \tau, \|T_n(x)\| \leq M_\varepsilon\|x\|] \\ &\geq \mathscr{P}[\|(T_n - T)(x)\| < \tau] + \mathscr{P}[\|T_n(x)\| \leq M_\varepsilon\|x\|] - 1 \\ &\geq \mathscr{P}[\|(T_n - T)(x)\| < \tau] + \varepsilon - (1 - \beta) - 1.\end{aligned}$$

Thus, letting $n \to \infty$ we obtain that

$$\mathscr{P}[\|T(x)\| \leq M_\varepsilon\|x\| + \tau] \geq \beta - (1 - \varepsilon), \quad \forall x \in X,$$

and letting $\tau \to 0$ we obtain that

$$\mathscr{P}[\|T(x)\| \leq M_\varepsilon\|x\|] \geq \beta - (1 - \varepsilon), \quad \forall x \in X,$$

so $\alpha(T) \geq \beta$, which is impossible. □

Given $\{T_n\}_{n\in\mathbb{N}}$ and $0 < \delta < 1$, we say that a linear random operator, T, is a *δ-limit* of $\{T_n\}_{n\in\mathbb{N}}$ if for all x in X

$$\underline{\lim}\mathscr{P}[\|T_n(x) - T(x)\| \leq \tau] \geq \delta, \quad \forall\tau > 0.$$

Obviously, *T is the stochastic pointwise limit of $\{T_n\}_{n\in\mathbb{N}}$ if and only if T is a δ-limit of $\{T_n\}_{n\in\mathbb{N}}$ for all δ in $]0, 1[$.*

The next corollary of the random Banach-Steinhaus theorem shows that, even if $\{T_n\}_{n\in\mathbb{N}}$ have an infinite number of different δ-limits, if δ is large enough ($\delta > \frac{2}{3}$) and the operators T_n are stochastically continuous, then every δ-limit, T, is probably continuous. In addition the greater δ, the greater $\alpha(T)$. Moreover, if T is the stochastic pointwise limit of $\{T_n\}_{n\in\mathbb{N}}$, then $\alpha(T) = 1$ (i.e. T is stochastically continuous).

Corollary 11. *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of stochastically continuous linear random operators, and let T be a linear random operator such that there exists $\delta > \frac{2}{3}$ satisfying that, for all x in X,*

$$\underline{\lim}\mathscr{P}[\|T_n(x) - T(x)\| \leq \tau] \geq \delta, \quad \forall\tau > 0.$$

Then T is probably continuous and $\alpha(T) \geq 3\delta - 2$.

Proof. Let $0 < \delta' < 3\delta - 2$. Since $\frac{2+\delta-\delta}{2} < \delta$, by hypothesis, for every x in X, there exists a natural number m, which depends on x, such that

$$\mathscr{P}[\|T_n(x) - T(x)\| \leq 1] > \frac{2 + \delta' - \delta}{2}, \quad \forall n \geq m,$$

so, there exists a constant N_x such that

$$\mathscr{P}[\|T_n(x)\| \leq N_x] > \frac{2 + \delta' - \delta}{2}, \quad \forall n \geq m.$$

In consequence, for all x in X, there exists $M_x > 0$ satisfying

$$\mathscr{P}[\|T_n(x)\| \leq M_x] > \frac{2 + \delta' - \delta}{2}, \quad \forall n \in \mathbb{N},$$

and, by the random Banach-Steinhaus theorem, we obtain a constant, $M_{\delta'}$, such that

$$\mathscr{P}[\|T_n(x)\| \leq M_{\delta'}\|x\|] \geq (1 + \delta' - \delta) - (1 - \alpha(T_n)), \quad \forall x \in X, \forall n \in \mathbb{N}.$$

Since T_n is stochastically continuous, by Theorem 2, $\alpha(T_n) = 1$ and thus

$$\mathscr{P}[\|T_n(x)\| \leq M_{\delta'}\|x\|] \geq (1 + \delta' - \delta), \quad \forall x \in X, \forall n \in \mathbb{N}.$$

Arguing as in the last corollary, for $\tau > 0$,

$$\mathscr{P}[\|T(x)\| \leq M_{\delta'}\|x\| + \tau] \geq \mathscr{P}[\|T_n(x) - T(x)\| \leq \tau] + \mathscr{P}[\|T_n(x)\| \leq M_{\delta'}\|x\|] - 1, \quad \forall n \in \mathbb{N},$$

so

$$\mathscr{P}[\|T(x)\| \leq M_{\delta'}\|x\| + \tau] \geq \underline{\lim}\mathscr{P}[\|T_n(x) - T(x)\| \leq \tau] + (1 + \delta' - \delta) - 1 \geq \delta'.$$

If $\tau \to 0$, then

$$\mathscr{P}[\|T(x)\| \leq M_{\delta'}\|x\|] \geq \delta', \quad \forall x \in X. \quad \square$$

If T is the pointwise limit of a sequence of linear random operators, then *the probability of T being continuous* (i.e., $\alpha(T)$) depends on how continuous the functionals of the sequence are (by Corollary 10) and how good the pointwise convergence is (by Corollary 11).

Let X, Y and Z be Banach spaces, B_X the unit ball of X and B_Y the unit ball of Y. Assuming that the concept of bilinear random operator (from $X \times Y$ to Z) is in the reader's mind, we establish the following corollary of the random Banach-Steinhaus theorem.

Corollary 12. *Let T be a bilinear random operator, from $X \times Y$ to Z, such that the operators $T_y(x) := T(x, y)$, from X to Z, and $T_x(y) := T(x, y)$, from Y to Z, are probably continuous for all y in Y and all x in X, respectively. Then T is probably jointly continuous and*

$$\alpha(T) \geq \max\{\alpha_X - 2(1 - \alpha_Y), \alpha_Y - 2(1 - \alpha_X)\}$$

where

$$\alpha_X = \inf\{\alpha(T_x): x \in B_X\} \quad \textit{and} \quad \alpha_Y = \inf\{\alpha(T_y): y \in B_Y\}.$$

Proof. We consider $0 < \delta < \alpha_Y$. For all y in Y, T_y is probably continuous, so there exists $M_y > 0$ such that

$$\mathscr{P}[\|T_y(x)\| \leq M_y] \geq \delta, \quad \forall x \in B_X.$$

Since $T_x(y) = T_y(x)$, this shows that the family $\{T_x: x \in B_X\}$ is probably pointwise bounded so, by the random Banach-Steinhaus theorem, there exists $M_\delta > 0$ such that

$$\mathscr{P}[\|T(x, y)\| \leq M_\delta] \geq (2\delta - 1) - (1 - \alpha_X), \quad \forall y \in B_Y, \forall x \in B_X.$$

Thus,

$$\alpha(T) \geq (2\alpha_Y - 1) - (1 - \alpha_X).$$

In the same way, for $0 < \delta' < \alpha_X$ there exists $M_{\delta'} > 0$ such that

$$\mathscr{P}[\|T(x, y)\| \leq M_{\delta'}] \geq (2\delta' - 1) - (1 - \alpha_Y), \quad \forall x \in B_X, \forall y \in B_Y.$$

Therefore

$$\alpha(T) \geq (2\alpha_X - 1) - (1 - \alpha_Y),$$

and the result is proved. □

Corollary 13. *If a bilinear random operator is separately stochastically continuous, then it is jointly stochastically continuous.*

References

1. A. T. Bharucha-Reid, *Random integral equations*, Academic Press, New York and London, 1972.
2. M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer-Verlag, New York, 1991.
3. A. V. Skorohod, *Random linear operators*, Reidel, Holland, 1984.
4. M. V. Velasco and A. R. Villena, *Continuity of random derivations*, Proc. Amer. Math. Soc. **123** (1995), 107–120.
5. ——, *A random closed graph theorem* (submitted).

6. A. R. Villena, *Stochastic continuity of random derivations on H^*-algebras*, Proc. Amer. Math. Soc. **123** (1995), 785–796.
7. A. Wilansky, *Modern methods in topological vector spaces*, McGraw-Hill, New York, 1978.

Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain

E-mail address: mvvelasco@ugr.es

E-mail address: avillena@ugr.es