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A NOTE ON THE GENERALIZED DUMBBELL PROBLEM

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(Communicated by Barbara Lee Keyfitz)

Dedicated to the memory of Peter Greenberg

ABSTRACT. This note is devoted to the calculation of the asymptotics of the small eigenvalues and corresponding eigenfunctions for the Laplace operator with Neumann boundary conditions on a domain obtained by adding several thin channels between given bounded domains.

This note will employ a useful lemma on quadratic forms to improve, with a simple proof, a recent result of Jimbo and Morita. This lemma was introduced by Helffer and Sjöstrand to study the tunnelling effect, then by Colin de Verdière in his work about stable multiplicity and also by myself to study the growth of multiplicity by adding handles (see also an illustration of these techniques in [CCdV]). One can find the following weak version in [A].

Lemma. *Let* $(q, \mathscr{D})$ *be a closed non-negative quadratic form in the Hilbert space* $(\mathscr{H}, <\,,\,>)$. *Define the associate norm* $\|f\|_1^2 = \|f\|^2 + q(f)$, *and the spectral projector* Π_I *for any interval* $I =]\alpha, \beta[$ *for which the boundary does not meet the spectrum* (q *will denote also the symmetric bilinear associated form*). *Then*

(i) *There exists a constant* $C > 0$, *which depends on* I, *such that, if* $f \in \mathscr{D}$ *and* $\lambda \in I$ *satisfy*

$$\forall g \in \mathscr{D} \quad |q(f, g) - \lambda < f, g > | \leq \delta \|f\| \, \|g\|_1,$$

then, if a *is less than the distance of* α *or* β *to the spectrum of* q,

$$\|\Pi_I(f) - f\|_1 = \|\Pi_{I^c}(f)\|_1 \leq \frac{C\delta}{a}\|f\| .$$

(ii) *Now suppose that the spectral space* $E(I)$ *relative to* I *has dimension* m *and that* $f_1, \dots, f_m$ *is an orthonormal family which satisfies, for all* j : $\|\Pi_{I^c}(f_j)\|_1 \leq \delta$. *Let* E *be the space spanned by the* f_j*'s. Then the distance between* E *and* $E(I)$ *has order* $O(\delta)$. *If moreover the quadratic form* $q_{|E}$ *is diagonal in this basis, with distinct eigenvalues* μ_k, *you can decompose* $E = \oplus^{\perp} E_k$, *with* E_k *the eigenspace related to* μ_k. *Let* $3a = \inf_{k \neq l} |\mu_k - \mu_l|$ *and*

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$I_k =]\mu_k - a, \mu_k + a[$; *then*

$$\operatorname{dist}(E_k, E(I_k)) = O\left(\frac{\delta^2}{a}\right)$$

and the eigenvalues of q *in* I_k *all admit the development* $\mu_k + O(\delta^2)$.

The distance between two vector spaces is defined as the distance between their respective orthogonal projectors; see [H].

1. The situation

In the generalized Dumbbell problem one studies the perturbation made by adding thin channels between bounded domains of $\mathbf{R}^n$. Let

$$\Omega(\varepsilon) = D_1 \cup \cdots \cup D_N \cup \Big(\bigcup_{1 \le i < j \le N} Q_{ij}(\varepsilon) \Big)$$

where D_j are bounded disjoint domains of $\mathbf{R}^n$, $n \geq 2$, such that $\bar{D}_i \cap \bar{D}_j = \emptyset$ if $i \neq j$ and $Q_{ij}(\varepsilon)$ is either empty or a channel between D_i and D_j isometric to the cylinder

$$\big\{(s, y) \in \mathbf{R}^n;\ -s_{ij} \le s \le s_{ij} \text{ and } \|y\| < \varepsilon \rho_{ij}(s)\big\}.$$

We assume that $\Omega(\varepsilon)$ is connected and that the function $\rho_{ij} \in C^1([-s_{ij}, s_{ij}])$ satisfies $\rho_{ij} > 0$. The channel $Q_{ij}(\varepsilon)$ is glued at $p_{ij} \in \partial D_i$ and $p_{ji} \in \partial D_j$, so we assume that there exists, in a neighbourood of $p_{ij} \in \mathbf{R}^n$, isometric coordinates $(x_1, x') \in U$ centered at p_{ij} such that $U \cap D_i = \{(x_1, x'),\ x_1 < 0\}$. We suppose finally that if $(i, j) \neq (i', j')$, then $Q_{ij}(\varepsilon) \cap Q_{i'j'}(\varepsilon) = \emptyset$ and that $\bar{D}_k \cap Q_{ij}(\varepsilon) \neq \emptyset$ only if $k = i$ or $k = j$. In this case, $D_i \cap Q_{ij}(\varepsilon)$ consists exactly of the points $(-s_{ij}, y) \in Q_{ij}(\varepsilon)$ indentified with the points $(0, y)$ in the neighbourood above.

Remark. Here we have constructed a C^1-manifold. This is sufficient to define the quadratic form, and thus the Laplace operator by polarization. If we want more regularity we have to assume more about the functions ρ_{ij}; see [JM].

The problem concerns the convergence of the spectrum and the eigenfunctions of the Laplace operator $\Delta^{\text{Neu}}(\Omega(\varepsilon))$ with Neumann boundary condition on $\Omega(\varepsilon)$, as $\varepsilon \to 0$. I recall the

Proposition (Jimbo and Morita). *The spectrum of* $\Delta^{\text{Neu}}(\Omega(\varepsilon))$ *converges to the union* (*with multiplicity*) *of the spectrums of* $\Delta^{\text{Neu}}(D_i)$ *and of the operator on the interval* $]-s_{ij}, s_{ij}[$ *defined by* $\frac{-1}{\rho_{ij}^{n-1}} \frac{\partial}{\partial s}(\rho_{ij}^{n-1} \frac{\partial}{\partial s})$ *and Dirichlet boundary conditions.*

Hence 0 has multiplicity N at the limit and $\Delta^{\text{Neu}}(\Omega(\varepsilon))$ has N eigenvalues $\mu_0(\varepsilon) = 0 < \mu_1(\varepsilon), \ldots, \mu_{N-1}(\varepsilon)$ which converge to 0. We call these eigenvalues the small eigenvalues. The main result of [JM] is to show that $\lim_{\varepsilon \to 0} \mu_k(\varepsilon)/\varepsilon^{n-1}$ exists and is equal to the k^{th} eigenvalue of a certain symmetric matrix.

In order to apply the lemma to the quadratic form $q(f) = \int_{\Omega(\varepsilon)} |df|^2$ with domain $H^1(\Omega(\varepsilon))$, let $E(\varepsilon)$ be the total eigenspace corresponding to the small eigenvalues. As test functions, we will take Φ_i defined by

$$\Phi_i = \begin{cases} |D_i|^{-1/2} & \text{on } D_i\,, \\ 0 & \text{on } D_j\,,\ i \neq j\,, \\ |D_i|^{-1/2} \frac{\int_s^{s_{ij}} \rho_{ij}^{1-n}(t)dt}{\int_{-s_{ij}}^{s_{ij}} \rho_{ij}^{1-n}(t)dt} & \text{on } Q_{ij} \neq \emptyset\,; \end{cases}$$

here $|D_i| = \mathrm{Vol}(D_i)$. We denote by E the space spanned by the Φ_i, $1 \le i \le N$. We have clearly $\Phi_i \in H^1(\Omega(\varepsilon))$ for all $\varepsilon > 0$.

Let τ_{n-1} be the volume of the unit sphere $\mathbf{S}^{n-1}$ and $\kappa_{ij} = \int_{-s_{ij}}^{s_{ij}} \rho_{ij}^{1-n}(t)dt$ if $Q_{ij} \neq \emptyset$ and $\kappa_{ij} = \infty$ in the other case.

Theorem. *For $n \ge 3$ the eigenspace $E(\varepsilon)$ converges to E in such a way that*

$$\mathrm{dist}(E(\varepsilon), E) = O(\varepsilon^{n/2}).$$

The eigenvalues satisfy $\mu_k(\varepsilon) = \tau_{n-1}\varepsilon^{n-1}\lambda_k + O(\varepsilon^n)$ where $\lambda_0 = 0 < \lambda_1 \le \cdots \le \lambda_{N-1}$ are the eigenvalues of the symmetric matrix B defined by $B_{ii} = \frac{1}{|D_i|}\sum_j \frac{1}{\kappa_{ij}}$ and for $i \neq j$, $B_{ij} = -\frac{1}{\sqrt{|D_i|\,|D_j|}}\frac{1}{\kappa_{ij}}$.

Moreover if the eigenvalues of B are all distinct (which is generically the case), and if U is an orthogonal basis such that $U^{-1}BU$ is diagonal, then the eigenfunction $\Phi_k(\varepsilon)$ satisfies

$$\Phi_k(\varepsilon) = \phi_k + O(\varepsilon)\,, \ \text{with } \phi_k = \sum_l U_{lk}\Phi_l.$$

In dimension 2 we have the same result with a remainder of order $O(\varepsilon^{1-\eta})$ for all $\eta > 0$.

Remark. Even when the matrix B has eigenvalues with multiplicity we can construct series of eigenfunctions which converge to a limit eigenfunction. When the eigenvalues of B are simple we can give a priori the limit eigenfunction. This is the content of our result. As a supplementary and general result we can give the order of the remainders and the distance to the eigenspace.

2. Proof of the theorem

In order to apply the lemma, we need the following more or less obvious estimates:

2.1. *The matrix $Gij = \langle\Phi_i\,, \Phi_j\rangle$ satisfies*

$$G = I + O(\varepsilon^{n-1}).$$

We can then define A to be a symmetric matrix such that $A^2 = G^{-1}$. The functions $\Psi_i = \sum_j A_{ij}\Phi_j$ are orthonormal and satisfy $\Psi_i = \Phi_i + O(\varepsilon^{n-1})$. We will apply the lemma directly to the Ψ_i's .

2.2. *The restricted quadratic form* $q_{|E}$ *is written in the* Ψ_i*'s as follows*:

$$q_{|E}(\Psi) = \tau_{n-1}\varepsilon^{n-1}B + O(\varepsilon^{2(n-1)}).$$

To see this, remark that $q(\Psi_i, \Psi_j) = \sum A_{ik}A_{jl}q(\Phi_k, \Phi_l)$ so that, in terms of matrices, $q(\Psi) = Aq(\Phi)A^t = q(\Phi) + \|q(\Phi)\|O(\varepsilon^{n-1})$. A simple calculation gives $q(\Phi) = \tau_{n-1}\varepsilon^{n-1}B$.

2.3. *There exists a constant* $C > 0$ *such that for all* $f \in E$ *and* $g \in H^1(\Omega(\varepsilon))$:

$$q(f, g) \leq C\varepsilon^{n/2}\|f\|\,\|g\|_1.$$

The first part of the theorem follows directly by applying the lemma with $\delta = \varepsilon^{n/2}$.

Proof of 2.3. It is sufficient to show this estimate for $f = \Phi_i$:

$$\begin{aligned} q(\Phi_i, g) &= \sum_j \int_{-s_{ij}}^{s_{ij}} \int_{|y|<\varepsilon\rho_{ij}} \frac{1}{\sqrt{|D_i|}\kappa_{ij}} \frac{\partial g}{\partial s}(\sigma, y)\rho_{ij}^{1-n}(\sigma)\, dy\, d\sigma \\ &= \sum_j \int_{-s_{ij}}^{s_{ij}} \int_{|y|<\varepsilon} \frac{1}{\sqrt{|D_i|}\kappa_{ij}} \frac{\partial g}{\partial s}(\sigma, \rho_{ij}(\sigma)y)\, dy\, d\sigma. \end{aligned}$$

Let $h(s, y) = g(s, \rho_{ij}(s)y)$, so that

$$\frac{\partial g}{\partial s}(\sigma, \rho_{ij}(\sigma)y) = \frac{\partial h}{\partial s}(\sigma, y) - \sum_k y_k\rho'_{ij}(\sigma)\frac{\partial g}{\partial y_k}(\sigma, \rho_{ij}(\sigma)y).$$

If we substitute this expression into $q(\Phi_i, g)$ we obtain two terms.

The first one is a boundary term:

$$\begin{aligned} &\sum_j \int_{-s_{ij}}^{s_{ij}} \int_{|y|<\varepsilon} \frac{1}{\sqrt{|D_i|}\kappa_{ij}} \frac{\partial h}{\partial s}(\sigma, y)\, dy\, d\sigma \\ &\quad = \frac{1}{\sqrt{|D_i|}\kappa_{ij}}\left(\int_{|y|<\varepsilon} g(s_{ij}, \rho_{ij}(s_{ij})y)dy - \int_{|y|<\varepsilon} g(-s_{ij}, \rho_{ij}(-s_{ij})y)dy\right), \end{aligned}$$

i.e. a term on the boundary of the D_i's. For any positive numbers p, q such that $1/p + 1/q = 1$ we have

$$\left|\int_{|y|<\varepsilon} g(s_{ij}, \rho_{ij}(s_{ij})y)dy\right| \leq \left(\tau_{n-1}\left(\frac{\varepsilon}{\rho_{ij}(s_{ij})}\right)^{n-1}\right)^{1/q}\left(\int_{\partial D_j}|g|^p\right)^{1/p}.$$

Sublemma. *There exists a constant* $c > 0$ *such that for all* $g \in H^1(\Omega(\varepsilon))$ *and for* $p = \frac{2n-2}{n-2}$

$$\left(\int_{\partial D_j}|g|^p\right)^{1/p} \leq c\|g\|_1.$$

This sublemma is a simple consequence of the trace theorem, which is true for $p > 1$:

$$\int_{\partial D} |g|^p \leq \text{const.}\|g\|_1 \sqrt{\int_D |g|^{2p-2}},$$

and the Sobolev imbedding theorem

$$\int_D |g|^{\frac{2n}{n-2}} \leq \text{const.}\|g\|_1^{\frac{2n}{n-2}}$$

(see [B] for this sort of technique).

Hence, if $p = \frac{2n-2}{n-2}$, then $\frac{1}{q} = 1 - \frac{1}{p} = 1 - \frac{n-2}{2n-2} = \frac{n}{2n-2}$ and we can bound the first term as announced.

The second term is easier: $y_k \rho'_{ij}(\sigma) = O(\varepsilon)$ and so

$$\left|\sum_j \int_{-s_{ij}}^{s_{ij}} \int_{|y|<\varepsilon} \frac{1}{\sqrt{|D_i|\kappa_{ij}}} \sum_k y_k \rho'_{ij}(\sigma) \frac{\partial g}{\partial y_k}(\sigma, \rho_{ij}(\sigma)y) dy d\sigma\right| \leq \text{const. } \varepsilon^{\frac{n+1}{2}} \|g\|_1$$

by the Cauchy-Schwartz inequality.

2.4. *In order to separate the eigenfunctions corresponding to different* λ_k, we have to look at the second part of the lemma: in our case, up to constant, $\delta = \varepsilon^{n/2}$ and $a = \varepsilon^{n-1}$ so $\delta^2/a = \varepsilon$.

2.5. *In dimension* 2 the trace theorem is always valid for $p > 1$ and the Sobolev imbedding theorem works also for any $p > 1$, hence we can take any q, $\frac{1}{q} \underset{<}{\to} 1$.

References

[A] C. Anné, *Fonctions propres sur des variétés avec des anses fines, application à la multiplicité*, Comm. Partial Differential Equations **15** (1990), 1617–1630.

[B] H. Brézis, *Analyse fonctionnelle*, Masson, Paris, 1983.

[CCdV] B. Colbois and Y. Colin de Verdière, *Sur la multiplicité de la première valeur propre d'une surface à courbure constante*, Comment. Math. Helv. **63** (1988), 194–208.

[CdV] Y. Colin de Verdière, *Sur la multiplicité de la première valeur propre non nulle du Laplacien*, Comment. Math. Helv. **61** (1986), 254–270.

[H] B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Springer-Verlag, Berlin, 1988.

[HS1] B. Helffer and J. Sjöstrand, *Multiples wells in the semi-classical limit* I, Comm. Partial Differential Equations **9** (1984), 337–408.

[JM] S. Jimbo and Y. Morita, *Remarks on the behavior of certain eigenvalues on a singular perturbed domain with several thin channels*, Comm. Partial Differential Equations **17** (1992), 523–552.

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