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INFINITE DIFFERENTIABILITY IN POLYNOMIALLY BOUNDED O-MINIMAL STRUCTURES

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ABSTRACT. Infinitely differentiable functions definable in a polynomially bounded o-minimal expansion $\mathfrak{R}$ of the ordered field of real numbers are shown to have some of the nice properties of real analytic functions. In particular, if a definable function $f : \mathbb{R}^n \to \mathbb{R}$ is C^N at $a \in \mathbb{R}^n$ for all $N \in \mathbb{N}$ and all partial derivatives of f vanish at a, then f vanishes identically on some open neighborhood of a. Combining this with the Abhyankar-Moh theorem on convergence of power series, it is shown that if $\mathfrak{R}$ is a polynomially bounded o-minimal expansion of the field of real numbers with restricted analytic functions, then all C^∞ functions definable in $\mathfrak{R}$ are real analytic, provided that this is true for all definable functions of one variable.

INTRODUCTION

Throughout this note, $\mathfrak{R}$ denotes a fixed (but arbitrary) expansion of the structure $(\mathbb{R}, +, \cdot)$ in a first-order language extending $\{+, \cdot\}$. **Definable** means first-order definable in $\mathfrak{R}$ with parameters from $\mathbb{R}$. A function $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$, is said to be definable if its graph is definable. Whenever convenient, we may assume that we deal with totally defined functions by setting a definable function equal to 0 off its domain of definition. We say that $\mathfrak{R}$ is **polynomially bounded** if for every definable function $f : \mathbb{R} \to \mathbb{R}$ there exists $N \in \mathbb{N}$ such $|f(t)| \le t^N$ for all sufficiently large positive t. We say that $\mathfrak{R}$ is **o-minimal** if the definable subsets of $\mathbb{R}$ are just the finite unions of intervals of all kinds, including singletons.

The structure $(\mathbb{R}, <, 0, 1, +, -, \cdot)$ is polynomially bounded and o-minimal (by Tarski-Seidenberg); the sets definable in this structure are precisely the semialgebraic sets. (See [BCR] for a thorough treatment of semialgebraic sets.)

A polynomially bounded o-minimal structure in which non-semialgebraic sets are definable, due to Denef and van den Dries (see [DD] and [D]), is the **ordered field of real numbers with restricted analytic functions**

$$\mathbb{R}_{\mathrm{an}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, (\tilde{f})_{f \in \mathbb{R}\{X, m\}, m \in \mathbb{N}}),$$

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where $\mathbb{R}\{X, m\}$ denotes the ring of all power series in $X_1, \dots, X_m$ over $\mathbb{R}$ that converge in a neighborhood of $[-1, 1]^m$, and where for each $f \in \mathbb{R}\{X, m\}$ we define $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$ by

$$\tilde{f}(x) := \begin{cases} f(x), & x \in [-1, 1]^m, \\ 0, & x \in \mathbb{R}^m \setminus [-1, 1]^m. \end{cases}$$

The sets definable in $\mathbb{R}_{\text{an}}$ are the finitely subanalytic sets introduced in [D]; these are locally just like subanalytic sets, but have nicer global properties and behave better from a logical viewpoint. (See [BM] for general facts about subanalytic sets.)

At present, the largest known polynomially bounded o-minimal expansion of $(\mathbb{R}, +, \cdot)$ is the structure $(\mathbb{R}_{\text{an}}, (x \mapsto x^r)_{r \in \mathbb{R}})$, where we set $x^r := 0$ for $x \leq 0$ (see [M2]). The class of sets definable in this structure properly contains the class of finitely subanalytic sets; by [D] or [M2], the function $x \mapsto x^r : (0, +\infty) \to \mathbb{R}$ is definable in $\mathbb{R}_{\text{an}}$ if and only if r is rational.

Polynomially bounded o-minimal expansions of $(\mathbb{R}, +, \cdot)$ are becoming increasingly important objects of study. In this note, we establish some basic differentiability properties of functions definable in such structures.

Uniform bounds on orders of vanishing

Before we can state the main result, we need some further definitions and notational conventions.

Let U be an open subset of $\mathbb{R}^n$ ($n \geq 1$), and let $f : U \to \mathbb{R}$ be given. We say that f is C^∞ at $a \in U$ if f is C^∞ on some open neighborhood of a, and that f is analytic at a if f is (real) analytic on an open neighborhood of a. We say that f is **weak-C^∞** at $a \in U$ if for every $N \in \mathbb{N}$, f is C^N at a (i.e., there exists an open neighborhood U_N of a, $U_N \subseteq U$, such that $f \restriction U_N$ is C^N). If f is C^N at some $a \in U$ and all partial derivatives of f of order less than or equal to N (including f itself) vanish at a, we say that f is N-**flat** at a. If f is N-flat at a for all $N \in \mathbb{N}$, then f is said to be **flat** at a.

Let $U \subseteq \mathbb{R}^n$ be a definable open set. Then $C^\infty_{\text{df}}(U)$ denotes the ring of definable functions $f : U \to \mathbb{R}$ that are C^∞ on U. (Note that f is C^∞ on U if and only if f is weak-C^∞ at each $a \in U$.)

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|$ denotes $\max\{|x_1|, \dots, |x_n|\}$.

Given a set $A \subseteq \mathbb{R}^{m+n}$ ($m, n \geq 1$) and $x \in \mathbb{R}^m$, we put $A_x := \{y \in \mathbb{R}^n : (x, y) \in A\}$. For a function $f : A \to \mathbb{R}$ and $x \in \mathbb{R}^m$, if $A_x \neq \varnothing$, then $f(x, -)$ denotes the function $y \mapsto f(x, y) : A_x \to \mathbb{R}$. For convenience, we also allow the possibility that $m = 0$, in which case the obvious interpretations apply.

We come now to the main technical result of this paper:

Theorem. *Assume that $\mathfrak{R}$ is polynomially bounded and o-minimal. Let $f : A \to \mathbb{R}$ be definable, with $A \subseteq \mathbb{R}^{m+n}$ ($m \geq 0$ and $n \geq 1$). Then there exists $N \in \mathbb{N}$ such that for all $(x, y) \in A$, if y is in the interior of A_x and $f(x, -)$ is N-flat at y, then $f(x, -)$ vanishes identically in a neighborhood of y.*

Note that in the special case that $m = 0$ and A is open, we have that for all $y \in A$, if f is flat at y, then f vanishes identically in a neighborhood of y.

We will require for the proof the following result from [M2]:

(∗) *Let* $g : B \times \mathbb{R} \to \mathbb{R}$ *be definable,* $B \subseteq \mathbb{R}^p$ $(p \geq 0)$. *Then there exist* $r_1, \ldots, r_l \in \mathbb{R}$ *such that for all* $b \in \mathbb{R}^p$, *either* $g(b, t) = 0$ *for all sufficiently large (depending on* b) *positive* t, *or* $\lim_{t\to+\infty} g(b, t)/t^{r_i} = c$, *for some* $i \in \{1, \ldots, l\}$ *and* $c \in \mathbb{R} \setminus \{0\}$.

Proof of the Theorem. Let $F : A \times (0, \infty) \to \mathbb{R}$ be the definable function given by

$$F(x, y, t) := \begin{cases} \max_{z\in A_x \& |y-z|=t} |f(x, z)| & \text{if } \{z \in A_x : |y - z| = t\} \neq \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Applying (∗) to $(x, y, t) \mapsto F(x, y, 1/t)$, there exist $r_1, \ldots, r_l \in \mathbb{R}$ such that for all $(x, y) \in A$, either $F(x, y, t) = 0$ for all sufficiently small (depending on (x, y)) positive t, or $\lim_{t\to 0^+} F(x, y, t)/t^{r_i} \in \mathbb{R}\setminus\{0\}$ for some $i \in \{1, \ldots, l\}$. Choose $N \in \mathbb{N}$ with $N > \max\{r_1, \ldots, r_l\}$. Suppose that $(x, y) \in A$ is such that y is in the interior of A_x and $f(x, -)$ is N-flat at y. By Taylor's formula, $|f(x, z)| = O(|y - z|^N)$ as $|y - z| \to 0^+$; i.e., $F(x, y, t) = O(t^N)$ as $t \to 0^+$. Since $N > \max\{r_1, \ldots, r_l\}$, we must have $\lim_{t\to 0^+} F(x, y, t)/t^{r_i} = 0$ for $i \in \{1, \ldots, l\}$. Thus, $F(x, y, t) = 0$ for all sufficiently small positive t. It follows then (from the definition of F) that $f(x, -)$ vanishes identically in a neighborhood of y. □

Remark. The assumption that $\mathfrak{R}$ be polynomially bounded is necessary. By [M1], if $\mathfrak{R}$ is o-minimal and not polynomially bounded, then the exponential function e^x is definable. Thus, the conclusion of the theorem fails by the classic counterexample

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Corollary 1. *Assume that* $\mathfrak{R}$ *is polynomially bounded and o-minimal. Let* $U \subseteq \mathbb{R}^n$ *be an open connected definable set.*

(1) *If* $f \in C^\infty_{\mathrm{df}}(U)$ *is flat at some* $a_0 \in U$, *then* $f = 0$; *i.e.,* $C^\infty_{\mathrm{df}}(U)$ *is a quasianalytic class.*
(2) $C^\infty_{\mathrm{df}}(U)$ *is an integral domain.*

Proof. (1) Consider the definable open set A consisting of all $a \in U$ such that $f \restriction V = 0$ for some open $V \subseteq U$ with $a \in V$. By the Theorem (with $m = 0$), $a_0 \in A$. Let $a \in \mathrm{Cl}(A) \cap U$. All partials of f are continuous on U and vanish identically on A. Then all partials of f vanish at a; i.e., f is flat at a. By the Theorem, $a \in A$. Thus, A is both open and closed in U, so $A = U$.

(2) Let $f, g \in C^\infty_{\mathrm{df}}(U)$ with $fg = 0$. If $g(a) \neq 0$ for some $a \in U$, then f vanishes identically in a neighborhood of a; hence $f = 0$ by (1). □

Definable germs

Given $a \in \mathbb{R}^n$ we define an equivalence relation $\sim$ on the set of real-valued functions whose domain contains a neighborhood of a by $f \sim g$ if there is a neighborhood V of a, $V \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(g)$, such that $f \restriction V = g \restriction V$. The equivalence classes are called **germs** at a. The equivalence classes of definable functions that are weak-C^∞ at a are called **definable weak-C^∞ germs** at a. These germs can be added and multiplied in the usual way and are easily seen to form a local ring with maximal ideal the germs vanishing at a. We denote it by $\mathscr{D}_a^{\mathrm{wk}}$. We also let $\mathscr{D}_a^{\infty}$ (respectively, $\mathscr{D}_a^{\omega}$) denote the local rings of definable C^∞ (respectively, definable analytic) germs at a. For $a = 0 \in \mathbb{R}^n$, we write $\mathscr{D}^{\mathrm{wk}}(n)$, $\mathscr{D}^{\infty}(n)$, and $\mathscr{D}^{\omega}(n)$ as appropriate. Clearly, $\mathscr{D}_a^{\omega} \subseteq \mathscr{D}_a^{\infty} \subseteq \mathscr{D}_a^{\mathrm{wk}}$ for all $a \in \mathbb{R}^n$.

Proposition 1. *If* $\mathfrak{R}$ *is a polynomially bounded o-minimal expansion of* $\mathbb{R}_{\mathrm{an}}$ *such that* $\mathscr{D}^{\mathrm{wk}}(1) = \mathscr{D}^{\omega}(1)$, *then* $\mathscr{D}^{\mathrm{wk}}(n) = \mathscr{D}^{\omega}(n)$ *for all* $n \geq 1$.

Proof. By induction on n. The base case holds by assumption, so suppose that the result holds for n. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be definable and weak-C^∞ at $0 \in \mathbb{R}^{n+1}$. Then for every $r \in \mathbb{R}$ the definable function $(x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n, rx_n) : \mathbb{R}^n \to \mathbb{R}$ is weak-C^∞ at $0 \in \mathbb{R}^n$. By the inductive assumption, each such function is analytic at $0 \in \mathbb{R}^n$. By Abhyankar-Moh [AM], the Taylor series of f at $0 \in \mathbb{R}^{n+1}$ then converges on some neighborhood U of $0 \in \mathbb{R}^{n+1}$ to an analytic function $g : U \to \mathbb{R}$. Now for some $r > 0$, $g \restriction [-r, r]^{n+1}$ is definable in $\mathbb{R}_{\mathrm{an}}$. By Corollary 1(1) we have $f \restriction (-r, r)^{n+1} = g \restriction (-r, r)^{n+1}$. □

Remark. Clearly, in the above one can replace "$\mathscr{D}^{\mathrm{wk}}$" and "weak-$C^\infty$" by "$\mathscr{D}^{\infty}$" and "$C^\infty$", respectively.

In [M2], it is shown that $\mathscr{D}^{\mathrm{wk}}(1) = \mathscr{D}^{\omega}(1)$ for $(\mathbb{R}_{\mathrm{an}}, (x \mapsto x^r)_{r \in \mathbb{R}})$. Thus, given any function $f : U \to \mathbb{R}$ definable in $(\mathbb{R}_{\mathrm{an}}, (x \mapsto x^r)_{r \in \mathbb{R}})$, U open in $\mathbb{R}^n$, if f is weak-C^∞ at $a \in U$, then f is analytic at a.

Corollary 2. *Let* $\mathfrak{R}$ *be polynomially bounded and o-minimal. Then the function* $T : \mathscr{D}^{\mathrm{wk}}(n) \to \mathbb{R}[[X_1, \dots, X_n]]$ *sending the germ at* 0 *of a definable function* $f : \mathbb{R}^n \to \mathbb{R}$, *weak-*$C^\infty$ *at* 0, *to its formal Taylor expansion at* 0, *is an injective ring homomorphism.*

Proof. That T is a ring homomorphism is routine. By the Theorem, the kernel of T is the germ of the zero map $0 : \mathbb{R}^n \to \mathbb{R}$. □

Corollary 2 also holds with "$\mathscr{D}^{\infty}(n)$" in place of "$\mathscr{D}^{\mathrm{wk}}(n)$". It thus follows that if $\mathfrak{R}$ is polynomially bounded and o-minimal, then $\mathscr{D}^{\mathrm{wk}}(n)$ and $\mathscr{D}^{\infty}(n)$ are integral domains. (Of course, this is true for $\mathscr{D}^{\omega}(n)$ without assumptions that $\mathfrak{R}$ be polynomially bounded or o-minimal.)

Proposition 2. *The maximal ideals of* $\mathscr{D}^{\mathrm{wk}}(n)$, $\mathscr{D}^{\infty}(n)$, *and* $\mathscr{D}^{\omega}(n)$ *are each generated by the germs at* 0 *of the coordinate functions* $x \mapsto x_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$.

Proof. We do only the case of $\mathscr{D}^{\mathrm{wk}}(n)$; the others are similar.

First, suppose that $f : U \times V \to \mathbb{R}$ is definable and C^{N+1} for some $N \in \mathbb{N}$, with $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}$ an open box neighborhood of $0 \in \mathbb{R}^{m+1}$, $m \geq 0$. Consider the definable function $g : U \times V \to \mathbb{R}$ with

$$g(x, y) = \begin{cases} \frac{f(x, y) - f(x, 0)}{y}, & y \neq 0, \\ \frac{\partial f}{\partial y}(x, 0), & y = 0. \end{cases}$$

Given $(x, y) \in U \times V$ we have

$$f(x, y) - f(x, 0) = \int_0^1 \frac{d}{dt}(f(x, ty))\, dt = y \int_0^1 \frac{\partial f}{\partial y}(x, ty)\, dt.$$

Thus, g is C^N on $U \times V$. Note that $f(x, y) = f(x, 0) + yg(x, y)$ for all $(x, y) \in U \times V$. Using this fact, an easy induction on n yields the result. □

A local ring R with maximal ideal M is called **Henselian** if given $P \in R[T]$ and $a \in R$ with $P(a) \in M$ and $P'(a)$ invertible, there exists $b \in R$ with $P(b) = 0$ and $a \equiv b \bmod M$. It is easy to see that the implicit function theorems (C^N, C^∞ and analytic versions) yield definable functions when the data are definable; thus $\mathscr{D}^{\mathrm{wk}}(n)$, $\mathscr{D}^\infty(n)$ and $\mathscr{D}^\omega(n)$ are Henselian rings.

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