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INDECOMPOSABLE COALGEBRAS, SIMPLE COMODULES, AND POINTED HOPF ALGEBRAS

SUSAN MONTGOMERY

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ABSTRACT. We prove that every coalgebra C is a direct sum of coalgebras in such a way that the summands correspond to the connected components of the Ext quiver of the simple comodules of C. This result is used to prove that every pointed Hopf algebra is a crossed product of a group over the indecomposable component of the identity element.

INTRODUCTION

A basic structure theorem for cocommutative coalgebras asserts that any such coalgebra C is a direct sum of its irreducible components; as a consequence, it can be shown that any pointed cocommutative Hopf algebra is a skew group ring of the group G of group-like elements of H over the irreducible component of the identity element. These results were proved independently by Cartier and Gabriel and by Kostant in the early 1960's; see [Di, S1]. This paper is concerned with versions of these results for arbitrary coalgebras and for arbitrary (pointed) Hopf algebras.

In fact much is already known about the coalgebra problem. In 1975 Kaplansky [K] showed that any coalgebra C is (uniquely) a direct sum of indecomposable ones; moreover when C is cocommutative, the indecomposable components are irreducible. In 1978 Shudo and Miyamoto [ShM] defined an equivalence relation on the set of simple subcoalgebras of C and showed that the equivalence classes correspond to the indecomposable summands. A weaker version of this equivalence relation was studied recently in [XF].

In this paper we first refine what is known about the indecomposable components of C by proving a dual Brauer-type theorem: each component corresponds to a connected component of Γ_C, the Ext quiver of the simple (right) comodules of C (Theorem 2.1). In fact Γ_C can be viewed as a directed graph whose vertices are the simple subcoalgebras of C; in this formulation it extends the equivalence relation of [ShM]. We give a direct proof of these facts, and so give an alternate proof of the results of [K] and [ShM]. We use only a few basic coalgebra properties (such as local finiteness) and a theorem of Brauer on finite-dimensional algebras.

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Secondly, we apply the coalgebra result and prove that for any pointed Hopf algebra H, there is a normal subgroup N of the group G of group-like elements such that H is a crossed product of G/N over the indecomposable component of the identity element of H (Theorem 3.2).

When H is pointed, Γ_H can be described completely if all the skew-primitive elements are known; see Examples 1.3–1.5.

One motivation for our use of Γ_C comes from a recent paper of Chin and Musson [CM], in which they study the duals of certain quantum groups. For a Noetherian Hopf algebra H, they wish to generalize the definition of the hyperalgebra H' of H. To do this, they begin with the maximal ideal $m_0 = \ker \varepsilon$ of H and consider the "clique" of cofinite maximal ideals m which are equivalent to m_0 via a sequence of other such ideals which are "linked" to m in the sense of non-commutative Noetheran ring theory (see [GW]). This clique of maximal ideals of H dualizes to a set $\mathscr{S}_0$ of simple subcoalgebras of $C = H^0$, with the links dualizing to a relationship among the simple subcoalgebras of C. The set $\mathscr{S}_0$ determines a subcoalgebra D of C which is the "new" hyperalgebra.

Here we begin with an arbitrary coalgebra C and define the analog of links directly on the simple subcoalgebras of C; this gives us the graph Γ_C as mentioned above. The set $\mathscr{S}_0$ of [CM] is then the connected component of $k\varepsilon$ in Γ_C.

It was also shown in [CM] that if $H = \mathscr{O}_q(SL(2))$, then $C = H^0$ is a crossed product over the "hyperalgebra" D. This crossed product result suggested to us that a similar result might be true more generally.

1. Simple subcoalgebras and the quiver Γ_C

We first review a few definitions; we follow [S1] and [M2, Chapter 5]. Let k be a field. C will denote a k-coalgebra with comultiplication $\Delta: C \to C \otimes C$ and counit $\varepsilon: C \to k$. A basic fact is that C is *locally finite* in the sense that any finite subset of elements of C lies in a finite-dimensional subcoalgebra D of C. C is *simple* if it has no proper subcoalgebras; equivalently the linear dual C^* is a finite-dimensional simple k-algebra. C is *irreducible* if it has a unique simple subcoalgebra.

For any C, the *group-like elements* in C are the set $G(C) = \{x \in C | x \neq 0$ and $\Delta x = x \otimes x\}$; necessarily $\varepsilon(x) = 1$ for $x \in G(C)$. Note that a simple subcoalgebra D of C is one-dimensional $\Leftrightarrow D = kx$ for some $x \in G(C)$. A coalgebra is *pointed* if all of its simple subcoalgebras are one-dimensional.

For $x, y \in G(C)$, the *x, y-primitive* elements in C are the set $P_{x,y}(C) = \{c \in C | \Delta c = c \otimes x + y \otimes c\}$; necessarily $\varepsilon(c) = 0$ for $c \in P_{x,y}(C)$. Note that $k(x-y) \in P_{x,y}(C)$; an x, y-primitive element c is *non-trivial* if $c \notin k(x-y)$. If $x = y = 1$, the $1, 1$-primitives are simply called primitive; otherwise they are called skew primitive.

We first define the quiver using the simple subcoalgebras of C. Recall from [S, 9.0] that the *wedge* of two subspaces D and E of C is defined to be

$$D \wedge E = \Delta^{-1}(C \otimes E + D \otimes C).$$

Note that if D and E are subcoalgebras, then it is always true that $D \wedge E \supseteq D + E$.

1.1 **Definition.** Let $\mathcal{S}$ be the set of simple subcoalgebras of a coalgebra C.

(1) The *quiver* Γ_C is given as follows:
(V) the vertices of Γ_C are the elements of $\mathcal{S}$;
(E) there exists an edge $S_1 \to S_2$ for $S_i \in \mathcal{S}$, $\Leftrightarrow S_2 \wedge S_1 \neq S_1 + S_2$.

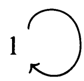
(2) C is called *link-indecomposable* (L.I.) if Γ_C is connected (as an undirected graph).

We will also say that S_1 and S_2 are *linked* if $S_1 \to S_2$ or $S_2 \to S_1$, and that S_1 and S_2 are *connected* (or $S_1 \sim S_2$) if they are in the same connected component of Γ_C.

1.2 *Remark.* When C is pointed and $x, y \in G(C)$, we will write $x \to y$ if $S_1 \to S_2$, where $S_1 = kx$ and $S_2 = ky$. It is known in this case that $x \to y \Leftrightarrow$ there exists a non-trivial x, y-primitive element. This follows by the Taft-Wilson theorem [M2, 5.4.1]; it is also not difficult to show directly. If C is a bialgebra and $x \to y$, then also $xz \to yz$ and $zx \to zy$, for all $z \in G(C)$. For, if $\Delta c = c \otimes x + y \otimes c$, then $\Delta(cz) = \Delta c \Delta z = cz \otimes xz + yz \otimes cz$, and so $xz \to xz$. Similarly $zx \to zy$ using zc.

We note that the connection between skew-primitive elements and links was observed in [CM].

1.3 **Example.** Let $C = U(g)$, the enveloping algebra of a non-zero Lie algebra g. Then $\mathcal{S} = \{k1\}$. Since $P_{1,1}(C) = g \neq 0$, Γ_c is given by a loop:



1.4 **Example.** Assume that k contains a primitive nth root of unity λ and let $C = H_{n^2}$, the Taft algebra of dimension n^2 [Tf]. This is a non-commutative, non-cocommutative Hopf algebra of dimension n^2 (the case $n = 2$ was defined by Sweedler). As an algebra,

$$H_{n^2} = k\langle g, x \mid g^n = 1, x^n = 0, xg = \lambda gx \rangle$$

with coalgebra structure given by $g \in G(H)$ and $x \in P_{1,g}(H)$. H_{n^2} is pointed with $G(H) = \{1, g, \dots, g^{n-1}\}$. Now $1 \to g$ since $\Delta x = x \otimes 1 + g \otimes x$; multiplying by g^k as in 1.2 we also have $g^k \to g^{k+1}$. Thus Γ_C is

$$1 \to g \to g^2 \to \cdots \to g^{n-1}.$$

1.5 **Example.** Consider $C = U_q(sl(2))$, for $q \in k^*$ not a root of 1, as described by Drinfeld and Jimbo; see [Dr]. That is, as an algebra

$$H = k\left\langle E, F, K, K^{-1} \mid KE = q^2 EK, KF = q^{-2} FK, EF - FE = \frac{K^2 - K^{-2}}{q^2 - q^{-2}} \right\rangle.$$

Its coalgebra structure is determined by $K \in G(H)$ and $E, F \in P_{K^{-1}, K}(H)$. It is known that H is pointed [M1, R] with $G(H) = \langle K \rangle$ and that the skew primitive elements are in the $kG(H)$-module spanned by 1, E, and F [M1, T1]. Moreover, for all n, EK^n and $FK^n \in P_{K^{n-1}, K^{n+1}}(H)$. Consequently Γ_C consists of two connected components:

$$\cdots \to K^{-3} \to K^{-1} \to K \to K^3 \to \cdots$$
$$\cdots \to K^{-2} \to 1 \to K^2 \to K^4 \to \cdots.$$

Thus $U_q(sl(2))$ is not link-indecomposable.

We now make the connection to extensions of simple comodules. Let M be any right C-comodule, with structure map $\Delta_M: M \to M \otimes C$, via $m \mapsto \sum m_{(0)} \otimes m_{(1)}$. M becomes a left C^*-module via $f \cdot m = \sum f(m_{(1)})m_{(0)}$, for all $f \in C^*$, $m \in M$, and two comodules M, N are isomorphic $\Leftrightarrow$ they are isomorphic as C^*-modules.

Now for each right comodule M there is a unique minimal subcoalgebra $C(M)$ of C such that $\Delta_M: M \to M \otimes C(M)$, that is, such that M is a $C(M)$-comodule ([S2, p. 326] or [A, p. 129]). If M is finite dimensional, so is $C(M)$, and if M is a simple comodule, then $C(M)$ is a simple subcoalgebra of C ([La, p. 354]: observe that by the minimality of $C(M)$, M is a faithful simple $C(M)^*$-module, and so $C(M)^*$ is a simple algebra). Moreover if $M \cong N$ are simple comodules, $C(M) = C(N)$. Conversely if S is a simple subcoalgebra of C, then S belongs to a simple right C-comodule M, unique up to isomorphism (namely, let M be a simple left S^*-module). Summarizing, we have the following well-known result.

1.6 **Lemma.** *There is a one-to-one correspondence between the set $\mathscr{M}$ of isomorphism classes of simple right C-comodules and the set $\mathscr{S}$ of simple subcoalgebras of C.*

Now let M, N be two simple right C-comodules. Then $\mathrm{Ext}^C(M, N) \neq 0 \Leftrightarrow$ there exists an exact sequence

$$0 \to N \to P \to M \to 0$$

where P is an indecomposable right C-comodule. The Ext *quiver* of $\mathscr{M}$ is the directed graph which has as vertices the elements of $\mathscr{M}$ and has an arrow from M to $N \Leftrightarrow \mathrm{Ext}^C(M, N) \neq 0$.

Recall that for a vector space V with dual space V^* and $U \subset V$, $U^\perp = \{f \in V^* | f(U) = 0\}$; similarly for $W \subset V^*$ we may define $W^\perp \subset V$.

1.7 **Theorem.** *Γ_C is isomorphic (as a directed graph) to the Ext quiver of simple (right) C-comodules.*

Proof. Lemma 1.6 gives a bijection between $\mathscr{S}$ and $\mathscr{M}$, and thus between the vertices of the graphs. To show the arrows correspond, we must show that for M, N in $\mathscr{M}$, $\mathrm{Ext}^C(M, N) \neq 0 \Leftrightarrow C(M) \to C(N)$ as in 1.1. To do this, we first prove it is true when C is finite dimensional.

Consider M and N as left C^*-modules as above and let $m = annM = C(M)^\perp$ and $n = annN = C(M)^\perp$; m and n are maximal ideals of C^*. By [S, 9.0.0], $(D^\perp E^\perp)^\perp = D \wedge E$, for any subspaces D and E of C. Thus

$$C(N) \wedge C(M) = (C(N)^\perp C(M)^\perp)^\perp = (nm)^\perp.$$

Also $C(M) + C(N) = m^\perp + n^\perp = (m \cap n)^\perp$. From 1.1, it follows that

$$C(M) \to C(N) \Leftrightarrow nm \neq n \cap m.$$

Now a standard argument for algebras shows that if n and m are maximal ideals of a finite-dimensional algebra A, and M and N are simple A/m- and A/n-modules respectively, then $nm \neq n \cap m \Leftrightarrow \mathrm{Ext}_A(M, N) \neq 0$. Since $A = C^*$ is finite dimensional, every A-module is a C-comodule, and conversely. Thus $C(M) \to C(N) \Leftrightarrow \mathrm{Ext}_{C^*}(M, N) \neq 0 \Leftrightarrow \mathrm{Ext}^C(M, N) \neq 0$. This proves the theorem in the finite-dimensional case.

We now show that in general, we may reduce to the finite-dimensional case. First assume that $\mathrm{Ext}^C(M, N) \neq 0$, and let P be an indecomposable C-comodule with $0 \to N \to P \to M \to 0$. Since M and N are finite dimensional, so is P. Thus $C(N)$, $C(P)$, and $C(M)$ all lie in some finite-dimensional subcoalgebra D of C. Moreover, since D-subcomodules are also C-subcomodules, N and M are simple D-comodules and P is an indecomposable D-comodule. Then $\mathrm{Ext}^D(M, N) \neq 0$, and so $C(M) \to C(N)$ in Γ_D since D is finite dimensional. By 1.1, clearly also $C(M) \to C(N)$ in Γ_C.

Conversely assume that $C(M) \to C(N)$ and choose $x \in C(N) \wedge C(M)$, $x \notin C(N)+C(M)$. Write $\Delta x \in \sum_i x_i \otimes C(M) + \sum_j C(N) \otimes y_j$ and let D be the (finite-dimensional) subcoalgebra of C generated by x, all x_i, all y_j, $C(N)$, and $C(M)$. Then $\Delta x \in D \otimes C(M) + C(N) \otimes D$ and so $C(M) \to C(N)$ in D. Since D is finite dimensional, the above argument shows $\mathrm{Ext}^D(M, N) \neq 0$. However, any D-comodule is also a C-comodule, and thus $\mathrm{Ext}^C(M, N) \neq 0$. □

We note that the finite-dimensional case of the above argument is essentially [CM, 1.1, Proposition]. Also the relationship $S + T = S \wedge T$ is mentioned in [XF], but only used for a criterion for determining when a coalgebra is a direct sum of irreducible ones.

At this point we compare Γ_C with the equivalence relation of Shudo and Miyamoto [ShM]. For $S, T \in \mathcal{S}$, they define $S \sim T \Leftrightarrow$ either $S = T$, or when $S \neq T$, there exists a finite chain $S = S_0, S_1, \ldots, S_n = T$, with all $S_i \in \mathcal{S}$, such that $S_i \wedge S_{i+1} \neq S_{i+1} \wedge S_i$, all $i = 0, \ldots, n-1$.

1.8 **Lemma.** *For $S, T \in \mathcal{S}$, S is connected to T in $\Gamma_C \Leftrightarrow S \sim T$ in the sense of* [ShM].

Proof. Using 1.1, it suffices to show that for $S \neq T$, $\{S \wedge T \neq S + T$ or $T \wedge S \neq S + T\} \Leftrightarrow S \wedge T \neq T \wedge S$. Equivalently,

$$\{S \wedge T = S + T \text{ and } T \wedge S = S + T\} \Leftrightarrow S \wedge T = T \wedge S.$$

$(\Rightarrow)$ is clear. To show $(\Leftarrow)$, choose $c \in S \wedge T = T \wedge S$. We must show $c \in S + T$. By the definition of wedge,

$$\begin{aligned}\Delta c &\in (C \otimes S + T \otimes C) \cap (C \otimes T + S \otimes C) \\ &= (S + T) \otimes (S + T).\end{aligned}$$

The equality is a standard vector-space argument using $S \cap T = (0)$. Since $c = (\mathrm{id} \otimes \varepsilon)\Delta c$, it follows that $c \in S + T$. □

Thus the [ShM] equivalence classes correspond to the connected components of Γ_C. However, by looking only at equivalence classes of $\mathcal{S}$, a lot of information is lost: the directions of the arrows, as well as the possibility of an arrow from S to itself, are not considered, nor is the connection to extensions of comodules.

2. The decomposition theorem for coalgebras

In this section, we prove that any coalgebra can be decomposed as a direct sum of indecomposable components, each of which corresponds to a connected component of Γ_C. Although this result could be obtained by combining work of [K], [ShM], and Lemma 1.8, we give instead a fairly short, direct proof, which follows the outline of the proof for the cocommutative case given in [S1].

We also use the classical theorem of Brauer, which states that a finite-dimensional algebra A is indecomposable $\Leftrightarrow$ the Ext quiver Γ_A of simple A-modules is connected (see for example, [P, p. 100]).

By a *link-indecomposable component* (L.I.C.), we mean a subcoalgebra D which is maximal with respect to Γ_D being connected.

2.1 **Theorem.** *Let C be any coalgebra. Then $C = \bigoplus_\alpha C_\alpha$, where the C_α are the link-indecomposable components of C.*

Proof. We proceed by a series of steps as in the cocommutative case [S, 8.0.7].

(1) It is a well-known fact [S,8.0.3], [M,5.6.2] that if $C = \sum_\alpha C_\alpha$ is any sum of coalgebras C_α, and D is any simple subcoalgebra of C, then D must lie in one of the C_α.

(2) We claim that if $\{C_\alpha\}$ is a family of L.I. subcoalgebras of C such that $\bigcap_\alpha C_\alpha \neq 0$, then also $\sum_\alpha C_\alpha$ is L.I.

For, $\bigcap_\alpha C_\alpha$ must contain a simple subcoalgebra, say D. Thus $D \subseteq C_\alpha$, for all α. Let E be any simple subcoalgebra of $\sum_\alpha C_\alpha$. By (1), $E \subseteq C_\beta$ for some β.

Thus both D and E are in C_β, which is L.I. Thus D and E are connected via a set of simples in $C_\beta \subseteq \sum_\alpha C_\alpha$. Similarly any simple F in $\sum_\alpha C_\alpha$ is connected to D, and thus to E. Thus $\sum_\alpha C_\alpha$ is L.I.

(3) Any L.I. subcoalgebra is contained in a unique L.I. component. For, let E be the sum of all L.I. subcoalgebras containing the given one; it is maximal (and unique) by construction, and is L.I. by (2).

(4) We claim that a sum of distinct L.I.C's is direct. For assume that $\{C_\alpha\}$ are distinct L.I.C's. If the sum is not direct, $C_\beta \cap (\sum_{\alpha\neq\beta} C_\alpha) \neq 0$ for some β. Let D be a simple subcoalgebra in this intersection. Since $D \subseteq \sum_{\alpha\neq\beta} C_\alpha$, it follows by (1) that $D \subseteq C_\gamma$ for some $\gamma \neq \beta$. Thus $0 \neq D \subseteq C_\beta \cap C_\gamma$. Applying (2), $C_\beta + C_\gamma$ is L.I. By assumption each C_α is maximal L.I., and thus $C_\beta = C_\beta + C_\gamma$, a contradiction. This proves the claim.

(5) By (3) and (4), it remains only to show that the sum of the distinct L.I.C.'s is all of C. By local finiteness, any $c \in C$ lies in a finite-dimensional subcoalgebra; thus we may assume that C is finite dimensional. Then $A = C^*$ is a finite-dimensional algebra, and so $A = \bigoplus_i A_i$ where the A_i are indecomposable algebras. Thus $C \cong A^* \cong \bigoplus_i A_i^*$. We are now done by Brauer's theorem, since $\Gamma_{A_i} \cong \Gamma_{A_i^*}$ as noted in Theorem 1.7. □

2.2 **Corollary.** *C is indecomposable $\Leftrightarrow \Gamma_C$ is connected.*

Proof. ($\Rightarrow$) If Γ_C is not connected, then C is not indecomposable by 2.1.

($\Leftarrow$) Assume Γ_C is connected, but that $C = D \oplus E$, for D, E proper subcoalgebras of C. By assumption Γ_C is connected; also by (1) in the proof of 2.1, every simple subcoalgebra of C lies in D or in E.

Thus we can find simple subcoalgebras $S \subseteq D$ and $T \subseteq E$ which are linked; say $S \to T$. Let $m = S^\perp$ and $n = T^\perp$ in C^*. Since $S \to T$, it follows by [S1, 9.0.0], as in 1.7, that $mn \neq m \cap n$, since $(mn)^\perp = S \wedge T \neq S + T = (m \cap n)^\perp$.

Now $C^* \cong D^* \oplus E^* \cong C^*/D^\perp \oplus C^*/E^\perp$. Under this isomorphism, since $m \supseteq D^\perp$, we may find an ideal I of $C^*/D^\perp$ such that $m = I \oplus C^*/E^\perp$. Similarly, since $n \supseteq E^{\perp}$, there is some ideal J of $C^*/E^{\perp}$ such that $n = C^*/D^{\perp} \oplus J$. It follows that $mn = m \cap n$, a contradiction. Thus C is indecomposable. □

2.3 *Remark.* If C is cocommutative and is link-indecomposable, then C is irreducible. For, assume C is not irreducible. Then C contains two simple subcoalgebras with $S_1 \to S_2$. If $x \in \Delta^{-1}(C \otimes S_1 + S_2 \otimes C)$, $x \notin S_1 + S_2$, consider the subcoalgebra C_1 generated by x, S_1, and S_2. Setting $m = S_1^{\perp}$ and $n = S_2^{\perp}$, we have (again as in 1.7) that $mn \neq m \cap n$. But C_1^* is a finite-dimensional commutative algebra, so is a finite direct sum of local rings. This is a contradiction. Thus we recover the classical result.

3. Pointed Hopf algebras

In this section we prove the crossed product decomposition mentioned in the introduction. We first need a known lemma.

3.1 **Lemma.** (1) *If C and D are pointed coalgebras, then $C \otimes D$ is pointed and $G(C \otimes D) = G(C) \otimes G(D)$.*

(2) *If $f\colon C \twoheadrightarrow D$ is a surjection of coalgebras and C is pointed, then D is pointed and $G(D) = f(G(C))$.*

Proof. (1) is [M2, 5.1.10] and (2) is [M2, 5.3.5]. □

In fact if C and D are pointed indecomposable, then $C \otimes D$ is also indecomposable. For, if $x \to y$ in C and $z \in G(D)$, then $x \otimes z \to y \otimes z$ as in 1.2, using tensor products. Similarly, if $z \to w$ in D, then $y \otimes z \to y \otimes w$, and thus $x \otimes z \to y \otimes w$, for all $x, y \in G(C)$ and $z, w \in G(D)$. Thus $G(C \otimes D)$ is connected and so $C \otimes D$ is indecomposable by 2.2. It is false, however, that images of pointed indecomposable Hopf algebras are indecomposable: for, $k\mathbf{Z}_2 = k1 + kg$ is an image of H_4, which is indecomposable by 1.4 and 2.2.

We let S denote the antipode of H. If $x \in G(H)$, then $Sx = x^{-1}$ and thus $G(H)$ is actually a group.

3.2 **Theorem.** *Let H be a pointed Hopf algebra, and set $G = G(H)$. For each $x \in G$, let $H_{(x)}$ denote the indecomposable component containing x. Then:*

(1) *$H_{(x)}H_{(y)} \subseteq H_{(xy)}$ and $SH_{(x)} \subseteq H_{(x^{-1})}$. In particular $H_{(1)}$ is a Hopf algebra.*

(2) *G acts on $H_{(1)}$ by $x \cdot h = xhx^{-1}$, for all $x \in G$, $h \in H_{(1)}$.*

(3) *The set $N = G(H_{(1)})$ is a normal subgroup of G.*

(4) *$H \cong H_{(1)}\#_{\sigma}k(G/N)$, a crossed product of $H_{(1)}$ with the quotient group G/N, with cocycle $\sigma\colon G/N \times G/N \to N$.*

Proof. (1) The argument generalizes those in [S1, §8.1] and [M2, 5.6.4]. If $x, y \in G$, then Lemma 3.1 implies that $H_{(x)} \otimes H_{(y)}$ is pointed indecomposable. Also multiplication $H_{(x)} \otimes H_{(y)} \to H_{(x)}H_{(y)}$ is a coalgebra surjection, and thus by 3.1 $H_{(x)}H_{(y)}$ is pointed with $G(H_{(x)}H_{(y)}) = \{zw | z \in G(H_{(x)}), w \in G(H_{(y)})\}$. Moreover a similar argument to the one after 3.1 shows that $G(H_{(x)}H_{(y)})$ is connected. Thus $H_{(x)}H_{(y)}$ is link-indecomposable; since it contains xy, it must be contained in $H_{(xy)}$. It follows that $(H_{(1)})^2 \subseteq H_{(1)}$ and so $H_{(1)}$ is a bialgebra. It remains to show that $SH_{(x)} \subseteq H_{(x^{-1})}$. Now S is bijective since H is pointed [M2, 5.2.11] and thus $S\colon H^{cop} \to H$ is a coalgebra isomorphism; here H^{cop} is

H with the opposite coalgebra structure. Thus $H_{(x)}$ indecomposable implies that $S(H^{cop}_{(x)})$ is indecomposable. Since $x^{-1} = Sx \in S(H^{cop}_{(x)})$, it follows that $S(H^{cop}_{(x)})$ is the indecomposable component containing x^{-1}. Thus $SH_{(x)} = H_{(x^{-1})}$.

(2) For each $x \in G$, the map $\phi_x\colon H \to H$ given by $h \mapsto xh$ is a coalgebra automorphism of H. Thus $\phi_x(H_{(1)}) = xH_{(1)}$ is the indecomposable component of H containing x, and so $xH_{(1)} = H_{(x)}$. Similarly $H_{(x)} = H_{(1)}x$. Consequently $xH_{(1)}x^{-1} = H_{(1)}$.

(3) Now $N = \{x \in G(H) | x \text{ is connected to } 1\}$ since $H_{(1)}$ is the component containing 1. N is a subgroup since $H_{(1)}$ is a Hopf algebra; however this is easy to see directly. For, $x \to y$ implies that $xz \to yz$ and $zx \to zy$ as in Remark 1.2. Thus if $x, y \in N$, so that $1 \sim x$ and $1 \sim y$, then $xy^{-1} \sim xy^{-1}y = x \sim 1$ and so $xy^{-1} \in N$. Thus N is a subgroup. Similarly if $1 \sim x$ and $z \in G$, then $1 \sim zxz^{-1}$ and so N is normal.

(4) The crossed product decomposition now follows from (2), (3), and Theorem 2.1. For, let T be a set of distinct coset representatives of N in G. Then $H = \bigoplus_{\overline{x} \in T} H_{(\overline{x})} = \bigoplus_{\overline{x} \in T} H_{(1)}\overline{x}$. Write $\overline{x}\,\overline{y} = \sigma(x, y)\overline{xy}$, where $\sigma(x, y) \in N$. Then if $h, k \in H_{(1)}$,

$$(h\overline{x})(k\overline{y}) = h\overline{x}k\overline{x}^{-1}\overline{x}\,\overline{y} = h(\overline{x} \cdot k)\sigma(x, y)\overline{xy}.$$

Thus $H \cong H_{(1)}\#_\sigma k(G/N)$, a crossed product. □

3.3 **Example.** We return to $U_q(sl(2))$, which we saw in 1.5 was not indecomposable. In this case the indecomposable component of 1 is

$$H_{(1)} = k\langle EK, FK, K^2, K^{-2}\rangle$$

with the same relations as before. Here $N = \langle K^2 \rangle$ and so $G/N = \langle \overline{K} \rangle \cong \mathbf{Z}_2$, where $\overline{K}$ is the coset KN. The cocycle $\sigma\colon G/N \times G/N \to N$ is given by $\sigma(\overline{K}, \overline{K}) = K^2$ and $\sigma(\overline{K}, \overline{1}) = \sigma(\overline{1}, \overline{K}) = \sigma(\overline{1}, \overline{1}) = 1$. Then

$$U_q(sl(2)) \cong H_{(1)}\#_\sigma k(G/N) \cong H_{(1)}\#_\sigma k\mathbf{Z}_2.$$

We note that if we set $E_1 = EK$, $F_1 = K^{-1}F$, and $K_1 = K^2$, then $H_{(1)} = k\langle E_1, F_1, K_1, K_1^{-1}\rangle$ is the "new version" of $U_q(sl(2))$ which is now used in [L], [DK], [Tk]. Lusztig remarks in [L] that the new version is used " to avoid certain irrelevant fourth roots of 1". Theorem 3.2 suggests that $H_{(1)}$ is the more fundamental object since its group-like elements are connected.

A similar decomposition holds for $U_q(sl(n))$, as its skew primitives are also spanned over $kG(H)$ by the E_i and F_i (M. Takeuchi, private communication, following [Tk]).

3.4 *Remark.* The $H_{(1)}$ in $U_q(sl(2))$ above was also used in [MS] in determining the possible actions of $U_q(sl(2))$ on the polynomial ring $\mathbf{C}[x]$. In that paper, it seemed easier to first deal with 1, K^2 -derivations and then to consider the action of K, rather than to begin with K^{-1}, K derivations. More generally, in considering actions of a pointed Hopf algebra H on a ring A, Theorem 3.2 suggests that one could first consider the indecomposable case $H_{(1)}$, and then use known results about group crossed products to deal with the action of G/N on A.

Added in Proof

J. A. Green shows in *Locally finite representations*, J. Algebra **41** (1976), 137–171, (1.6b), that the indecomposable components of a coalgebra are the "blocks" with respect to an equivalence relation on the simple comodules using their injective covers. Although "adjacency" of the simple comodules may be different than in [ShM] (and in this paper), the equivalence classes are the same.

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Department of Mathematics, University of Southern California, Los Angeles, California 90089-1113

E-mail address: `smontgom@mtha.usc.edu`