PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 8, August 1995

THE LAVRENTIEV PHENOMENON AND THE OBSTACLE PROBLEM FOR THE DIRICHLET INTEGRAL

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(Communicated by Albert Baernstein II)

ABSTRACT. It is shown that the Lavrentiev phenomenon can occur for the Dirichlet integral, when the obstacle problem is considered.

In 1926 M. Lavrentiev [L] observed that the regularity of the admissible functions for the minimization of certain variational integrals is decisive. For example, the minimum of the integral

$$I(y) = \int_0^1 (x - y^3(x))^2 y'(x)^6 \, dx$$

taken among all continuous functions with boundary values $y(0) = 0$, $y(1) = 1$, is zero, if all absolutely continuous functions with summable derivative are admissible. However, if the class of admissible functions is further restricted to those with bounded first derivative, then the infimum of the integral is strictly positive. In this example (essentially given in [M]) the explanation of the Lavrentiev phenomenon is transparent: the natural minimizer $y(x) = x^{1/3}$ does not have a bounded derivative and so it is ruled out in the second case. For another example see [BM]. (See [BuM] for a good introduction to the Lavrentiev phenomenon.)

The variational integrals are somewhat artificial in examples like this one. The aim of this note is to show that essentially the same phenomenon occurs for the ordinary Dirichlet integral in two or more dimensions when so-called obstacle problems are considered. This has the striking consequence that one is not always able to detect the minimum of the Dirichlet integral by using the usual finite element methods. More precisely, in the obstacle problem the admissible functions φ for the Dirichlet integral

$$\int |\nabla \varphi|^2 \, dx$$

are forced to live above some given function u, called the obstacle. Needless to say, pathological effects can be caused by choosing extremely unsuited obstacles. However, our point is that the Lavrentiev phenomenon can occur already for an obstacle that is a bounded superharmonic function, harmonic outside a set of measure zero.

Received by the editors December 6, 1993.
1991 *Mathematics Subject Classification.* Primary 49J30, 35J85, 35A35, 31B35.

Theorem. *Let Ω be a bounded open set in $\mathbf{R}^n$. There exists a bounded superharmonic function u in Ω such that*

$$\int_\Omega |\nabla u|^2\,dx < \inf \int_\Omega |\nabla \varphi|^2\,dx,$$

where the infimum is taken over all continuous functions $\varphi \in H^1(\Omega)$ such that $\varphi - u \in H_0^1(\Omega)$ and $\varphi \ge u$ in Ω. Moreover, u can be chosen to be harmonic in $\Omega \setminus C$, where C is a compact set of measure zero.

Before proving this remarkable fact, we shall recall some properties of superharmonic functions and then reformulate the minimization problem.

Let Ω be a domain in $\mathbf{R}^n$. A superharmonic function $u: \Omega \to (-\infty, \infty]$ is defined as a lower semicontinuous function satisfying the comparison principle with respect to harmonic functions: if h is harmonic in a subdomain D of Ω, continuous on $\overline{D}$, and $h \le u$ on the boundary of D, then necessarily $h \le u$ in D (or equivalently, u is a lower semicontinuous function whose mean value in any closed ball B in Ω does not exceed the value of u at the center of B). It is well known and easily follows from the Riesz decomposition theorem that the distributional (or Sobolev) derivative ∇u exists almost everywhere and $\nabla u \in L^p_{\mathrm{loc}}(\Omega)$ whenever $0 < p < n/(n-1)$. Moreover, if u is bounded, then $\nabla u \in L^2_{\mathrm{loc}}(\Omega)$, i.e., the Dirichlet integral of u evaluated over compact subsets is finite. Furthermore, any bounded superharmonic function u has a minimizing energy property. Namely,

$$\int_{\operatorname{supp}\varphi} |\nabla u|^2\,dx \le \int_{\operatorname{supp}\varphi} |\nabla(u+\varphi)|^2\,dx$$

whenever $\varphi \in C_0^\infty(\Omega)$ and $\varphi \ge 0$. Here $\operatorname{supp}\varphi$, the support of φ, is the closure of the set, where $\varphi(x) \ne 0$. The smoothness of φ does not yet play any essential role. Indeed, we could allow any nonnegative φ in the Sobolev space $H_0^1(\Omega)$ provided φ has compact support (or any nonnegative $\varphi \in H_0^1(\Omega)$ if the Dirichlet integral of u is finite).

Let us now fix a bounded superharmonic function u in a bounded domain Ω with

$$\int_\Omega |\nabla u|^2\,dx < \infty.$$

Then

$$\int_\Omega |\nabla u|^2\,dx \le \int_\Omega |\nabla v|^2\,dx$$

whenever $v \ge u$, $v \in H^1(\Omega)$, and $v - u \in H_0^1(\Omega)$. Note that u itself belongs to this class of admissible functions, so that u is the minimizing function. There are no other minimizing functions in this class.

Consider now the problem of minimizing the Dirichlet integral among all *continuous* functions $\varphi \ge u$, $\varphi \in H^1(\Omega)$, and $\varphi - u \in H_0^1(\Omega)$. Suppose that this class of admissible functions is not empty. This is certainly the case if u is continuous near the boundary of Ω. Via a minimizing sequence $\varphi_1, \varphi_2, \varphi_3, \dots$, one can show that there exists a function $w \in H^1(\Omega)$, $w \ge u$, $w - u \in H_0^1(\Omega)$, such that

$$\int_\Omega |\nabla w|^2\,dx = \inf_\varphi \int_\Omega |\nabla \varphi|^2\,dx.$$

From the parallelogram law

$$\int \left| \frac{\nabla\varphi - \nabla\tilde{\varphi}}{2} \right|^2 dx + \int \left| \frac{\nabla\varphi + \nabla\tilde{\varphi}}{2} \right|^2 dx = \frac{\int |\nabla\varphi|^2\, dx + \int |\nabla\tilde{\varphi}|^2\, dx}{2}$$

it easily follows that all minimizing sequences have the same limit, i.e. the minimizer w is unique. Moreover, w can be redefined in a set of measure zero so that it becomes superharmonic. This can be seen directly, but it is more easily deduced from the equivalent characterization below.

Next, consider all *continuous* superharmonic functions $\psi \in H^1(\Omega)$ such that $\psi \geq u$ in Ω. Take the pointwise infimum

$$v_1(x) = \inf_{\psi} \psi(x).$$

Observe that v_1 is upper semicontinuous, for it is a pointwise minimum of continuous functions. Furthermore, the lower semicontinuous function

$$v(x) = \liminf_{y\to x} v_1(y)$$

is superharmonic and $v \leq v_1$. Indeed, it is immediate that v obeys the comparison principle, and $v \leq v_1$ since v_1 is upper semicontinuous. We shall also need the fact that $v = v_1$ except possibly on a set of capacity zero [D, 1.VI.1., p. 70]. Note also that $v \geq u$ since u itself is lower semicontinuous by definition.

We claim that $v = w$ a.e. (but not that they coincide with u!).

To establish it, we take any minimizing sequences $\varphi_1, \varphi_2, \varphi_3, \dots$ for w. Replace each φ_k by the solution ψ_k to the obstacle problem

$$\int_\Omega |\nabla\psi_k|^2\, dx = \min_{\varphi} \int_\Omega |\nabla\varphi|^2\, dx,$$

where $\varphi \in H^1(\Omega)$ is continuous in Ω, $\varphi \geq \varphi_k$, and φ has the same boundary values as φ_k, i.e., $\varphi_k - \varphi \in H_0^1(\Omega)$. It is well known that the unique minimizer is a continuous superharmonic function and $\psi_k \geq \varphi_k \geq u$ (cf. [CK]). Since

$$\int_\Omega |\nabla\psi_k|^2\, dx \leq \int_\Omega |\nabla\varphi_k|^2\, dx,$$

the sequence $\psi_1, \psi_2, \psi_3, \dots$ is also minimizing. In particular, it has a subsequence ψ_k that converges to w a.e. in Ω. Replace any ψ in the definition for v by $\min(\psi, \psi_k)$. Then $\min(\psi, \psi_k) \geq v_1 \geq v$, so that $w \geq \min(\psi, w) \geq v$ a.e. in Ω.

The opposite inequality follows by noting that the sequence $\min(\psi_k, \psi)$ is again minimizing for w, because of the energy minimizing property of the superharmonic function $\min(\psi_k, \psi)$. Thus

$$w = \lim_{k\to\infty} \min(\psi_k, \psi) \leq \psi$$

a.e. in Ω, and hence $w \leq v_1$. We conclude that $w \leq v$ a.e., for v equals v_1 almost everywhere (cf. [D, 1.III.3., p. 37] or [LS, p. 228]). So we have that $w = v$ almost everywhere.

Our next step is to show that v is continuous except possibly on a set of capacity zero. To this end, let x_0 be any point in Ω such that

$$v_1(x_0) = v(x_0).$$

Since v is lower semicontinuous, we have that

$$\liminf_{x \to x_0} v(x) \geq v(x_0).$$

On the other hand, v_1 is upper semicontinuous as the infimum of continuous functions. Hence, because $v \leq v_1$, we obtain

$$\limsup_{x \to x_0} v(x) \leq \limsup_{x \to x_0} v_1(x) \leq v_1(x_0) = v(x_0).$$

Thus v is continuous at any point, where v coincides with v_1. Because this happens except possibly on a set of capacity zero, our desired claim follows.

We conclude the proof of the theorem by constructing a bounded superharmonic function u that is discontinuous in a set of positive capacity. Then $u < v = w$ in a set of positive measure, for any two superharmonic functions that are equal a.e. coincide everywhere according to Brelot's theorem (see [B, II.5]). Since

$$\int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega |\nabla w|^2 \, dx$$

by the minimizing property of the superharmonic functions, the theorem then follows from the uniqueness of the minimizer.

Thus is remains to construct u. The existence of such a function is known to the experts but, for the reader's convenience, we present a simple construction of a bounded superharmonic function u which is discontinuous in a set of positive capacity and harmonic outside a compact set of measure zero. To this end we fix a closed ball B in Ω. Let $Q = \{q_1, q_2, q_3, \dots\}$ be an enumeration of the points in B whose coordinates all are rational. For $j = 1, 2, \dots$, let

$$g_j(x) = \begin{cases} c_j |x - q_j|^{2-n} & \text{if } n \geq 3, \\ -c_j \log c|x - q_j| & \text{if } n = 2 \end{cases}$$

be a nonnegative fundamental solution of the Laplacian in Ω with pole at q_j, where $c_1, c_2, \dots$ are positive constants such that the function

$$g = \sum_{j=1}^{\infty} g_j$$

is finite at a fixed point x_0. Then g is a nonnegative superharmonic function in Ω. Furthermore, since the function $f(t) = t/(t+1)$ is concave and increasing on $(0, \infty]$, we have that the function

$$\tau(x) = \frac{g(x)}{1 + g(x)}$$

is superharmonic in Ω and $0 \leq \tau \leq 1$ [D, 1.II.9., p. 23]. Moreover, the set

$$E = \{x \in \Omega \colon \tau(x) = 1\} = \{x \in \Omega \colon g(x) = \infty\}$$

is of capacity zero, though it is a dense subset of B, for $Q \subsetneq E$. We conclude that τ is discontinuous at each point in $B \setminus E$, which has positive measure. By what we proved earlier, the function τ would serve as an obstacle for which the Lavrentiev phenomenon occurs. However, we shall modify τ further so that the resulting function u will be even harmonic outside a set of measure zero.

For this, we assume, as we may, that 0 is the center of B. Let

$$C = \{x = (x_1, x_2, \dots, x_n) \in B: x_1 = 0\}.$$

Then C is a compact set in Ω of measure zero, yet of positive capacity. If

$$\Phi(x) = \inf\{s(x): s \text{ superharmonic in } \Omega, s \geq 0, \text{ and } s \geq \tau \text{ on } C\},$$

then

$$u(x) = \liminf_{y \to x} \Phi(y)$$

is the function we are looking for. Indeed, u is superharmonic in Ω, harmonic in $\Omega \backslash C$, and $0 \leq u \leq 1$ (see [D, 1.VI.3, p. 73]). Our final task is to establish that u is discontinuous in a set of positive capacity.

To reach this conclusion, let $z \in C \backslash E$. Since E is of capacity zero, it is enough to show that u is not continuous at z. Let $2\delta = 1 - \tau(z) > 0$. Then $u(z) \leq \tau(z) = 1 - 2\delta$. The intersection of the open set

$$\{x \in \Omega: \tau(x) > 1 - \delta\}$$

with C is dense in C, so that for each $r > 0$ the set

$$U_r = \{x \in B(z, r): \tau(x) > 1 - \delta\} \cap C$$

is a nonempty, relatively open subset of C. Therefore, the $(n-1)$ measure, and hence the capacity of U_r, is positive. Since $u = \tau$ on C except possibly on a set of capacity zero, we find for each $r > 0$ a point $x \in U_r$ such that

$$u(x) = \tau(x) > 1 - \delta \geq u(z) + \delta.$$

Hence u cannot be continuous at z. This completes the proof of the theorem.

Let us finally remark that the method of proof above does not heavily rely on the properties of the Dirichlet integral. Indeed, a similar reasoning reveals that the Lavrentiev phenomenon occurs also for obstacle problems connected with variational integrals akin to

$$\int |\nabla \varphi|^p \, dx,$$

where $1 < p \leq n$.

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