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ON THE NUMBERS OF 2-WEIGHTS, UNIPOTENT CONJUGACY CLASSES, AND IRREDUCIBLE BRAUER 2-CHARACTERS OF FINITE CLASSICAL GROUPS

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ABSTRACT. A closed expression for the number of unipotent conjugacy classes of a classical group G is given and Alperin's weight conjecture is confirmed for G globally and for a symplectic or odd-dimensional orthogonal group block by block.

INTRODUCTION

Let G be a classical symplectic or orthogonal group defined over a field of odd characteristic. A 2-block B of G is labelled by a semisimple $2'$-element s^* in G by results of Cabanes and Enguehard [7] and the first author [3]. Let $\mathscr{W}(B)$ be the number of 2-weights in G associated with B and $\mathscr{U}_G(s^*)$ the number of unipotent conjugacy classes of $C_G(s^*)$, and let $l(B)$ be the number of irreducible Brauer 2-characters in B. In this paper we prove that $\mathscr{W}(B) = \mathscr{U}_G(s^*) = l(B)$ for each block B of a symplectic or odd-dimensional orthogonal group G. In addition, $\mathscr{W}(B) = \mathscr{U}_G(s^*)$ and $\mathscr{U}_{G_0}(s^*) = l(B')$ when G is an even-dimensional orthogonal group, where G_0 is the special orthogonal group, $\mathscr{U}_{G_0}(s^*)$ is the number of unipotent conjugacy classes in $C_{G_0}(s^*)$, and B' is a block of G_0 covered by B. In the latter case, we could not get the equation $\mathscr{U}_G(s^*) = l(B)$ because we do not know how to get $l(B)$. We give as corollaries a closed expression for the number of unipotent conjugacy classes of G, and get an affirmative answer for Alperin's weight conjecture globally for G and block by block for a symplectic or odd-dimensional orthogonal group G. Notice that the three numbers $\mathscr{W}(B)$, $\mathscr{U}_G(s^*)$, and $l(B)$ are also the same for a 2-block B of a general linear or unitary group by results of [1–2], [4], and [10].

In §1 we use the generating function given by Wall [14] for unipotent conjugacy classes of a symplectic or orthogonal group to show that $\mathscr{W}(B) = \mathscr{U}_G(s^*)$, and we give a closed formula for the number of unipotent conjugacy classes. In §2 we use the results of Broué [5] to show that $\mathscr{U}_{G_0}(s^*) = l(B')$.

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1. The numbers of weights for 2-blocks

Let $\mathbb{F}_q$ be a field of q elements with odd characteristic, and let V be a non-degenerate finite-dimensional symplectic or orthogonal space over $\mathbb{F}_q$. In addition, let $I(V)$ be the group of all isometries of V and $I_0(V)$ the subgroup of $I(V)$ of isometries of determinant 1. Thus $I(V) = I_0(V)$ if V is symplectic. Let $G = I_0(V)$, and let G^* be the dual group of G. If s is a semisimple element of G^*, then let (s) be the conjugacy class of G^* containing s, and let $\mathscr{E}(G, (s))$ be the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with (s) (see [6, p. 57]). If s is a semisimple $2'$-element of G^*, let

$$\mathscr{E}_2(G, (s)) = \bigcup_u \mathscr{E}(G, (su)),$$

where u runs over all 2-elements of $C_{G^*}(s)$. By [7, Theorem 13] and [5, Theorem 3.2], $\mathscr{E}_2(G, (s))$ is a 2-block, so that a 2-block B of $I(V)$ covers a 2-block $\mathscr{E}_2(G, (s))$. By [3, (5B)(c)], B covers another block $\mathscr{E}_2(G, (s'))$ if and only if s and s' are conjugate in $I(V^*)$, where V^* is the underlying space of G^*. For each semisimple $2'$-element s in $I_0(V)^*$ a dual element s^* in $I_0(V)$ of s is defined by [3, (4A)] and s^* is determined uniquely by s up to conjugacy in $I(V)$. We shall say that s^* is a *semisimple label* of B. Thus a semisimple label of B is determined uniquely up to conjugacy in $I(V)$. In this section, we shall show that the number of weights for a 2-block of $I(V)$ with semisimple label s^* is the number $\mathscr{U}_{I(V)}(s^*)$ of unipotent conjugacy classes of $C_{I(V)}(s^*)$. In particular, we shall get a closed formula for the number of unipotent conjugacy classes of $I(V)$.

First of all, we consider the symplectic group $\mathrm{Sp}(V)$. We shall need the following lemma.

Lemma (1A). *The following identity holds*:

$$\prod_{j=1}^{\infty}(1 - t^{2j}) \prod_{j=1}^{\infty}(1 + t^j) = \sum_{i=1}^{\infty} t^{\frac{i(i-1)}{2}}. \tag{1.1}$$

Proof. If $C(2, m)$ is the number of 2-cores of rank m, then

$$C(2, m) = \begin{cases} 1 & \text{if } m = \frac{i(i-1)}{2} \text{ for some } i \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.2}$$

where $\mathbb{N}$ is the set of all natural numbers. Now let $P(t)$ and $F_2(t)$ be the generating functions of partitions and 2-cores, respectively. Then

$$P(t) = \prod_{i=1}^{\infty} \frac{1}{1 - t^i} \quad \text{and} \quad F_2(t) = \sum_{i=1}^{\infty} t^{\frac{i(i-1)}{2}}, \tag{1.3}$$

and by [12, Proposition 3.3]

$$F_2(t) = (P(t^2))^{-2} P(t), \tag{1.4}$$

therefore

$$F_2(t) = \prod_{j=1}^{\infty}(1 - t^{2j})^2 \prod_{i=1}^{\infty} \frac{1}{1 - t^i} = \prod_{j=1}^{\infty}(1 - t^{2j}) \prod_{i=1}^{\infty}(1 + t^i).$$

Thus (1.1) holds.

As a consequence of (1A) we have the following corollary.

Corollary (1B). *Let $\pi_o(m)$ and $\pi_e(m)$ be the number of odd and even partitions of rank m, respectively, where an odd partition is a partition with odd parts and an even partition is a partition with even parts. Then*

$$\pi_o(m) = \sum_{j=1}^{m} \pi_e(j)C(2, m-j) = \sum_{i=1}^{m} \pi_e(m-i)C(2, i), \tag{1.5}$$

where $C(2, l)$ is the number of 2-cores of rank l given by (1.2).

Proof. Let $g_o(t)$ and $g_e(t)$ be the generating functions of odd and even partitions, respectively. Then

$$g_o(t) = \prod_{l=1}^{\infty}(1+t^l) \quad \text{and} \quad g_e(t) = \prod_{l=1}^{\infty} \frac{1}{1-t^{2l}}.$$

By (1A) $g_o(t) = g_e(t)F_2(t)$, and so (1.5) holds.

Proposition (1C). *Let G be the symplectic group $\mathrm{Sp}(V) = \mathrm{Sp}(2n, q)$ and B_0 the principal 2-block of G, and let $\mathscr{W}(B_0)$ be the number of B_0-weights. Then $\mathscr{W}(B_0)$ is the number $\mathscr{U}_G(1)$ of unipotent conjugacy classes of G. In particular,*

$$\mathscr{U}_G(1) = \sum_{\kappa} f_{X-1,\kappa},$$

where κ runs over all 2-cores with $|\kappa| \le n$ and $f_{X-1,\kappa}$ is the number of pairs (λ_1, λ_2) of partitions λ_i such that $|\lambda_1| + |\lambda_2| = n - |\kappa|$.

Proof. Let

$$f_{X-1} = \sum_{\kappa} f_{X-1,\kappa}, \tag{1.6}$$

where κ runs over all 2-cores with $|\kappa| \le n$. By [3, (6D)(d)] $\mathscr{W}(B_0) = f_{X-1}$.

Let $k(m, l)$ be the number of m-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of partitions λ_i such that

$$\sum_{i=1}^{m} |\lambda_i| = l.$$

Then by [13, p. 237],

$$P(t)^m = \sum_{l\ge 0} k(m, l)t^l. \tag{1.7}$$

The generating function of $\mathscr{W}(B_0)$ is

$$f(t) = \left(\sum_{l=0}^{\infty} k(2, l)t^{2l}\right) F_2(t^2) = \left(\prod_{i=1}^{\infty} \frac{1}{1-t^{2i}}\right)^2 F_2(t^2),$$

where $F_2(t)$ is the generating function of 2-cores given by (1.3). By [14, p. 38], the generating function of the number of unipotent conjugacy classes in G is

$$F_-(t) = \prod_{i=1}^{\infty} \frac{(1+t^{2i})^2}{1-t^{2i}}.$$

Now set $x = t^2$. It suffices to show that

$$\prod_{i=1}^{\infty} \frac{(1+x^i)^2}{(1-x^i)} = \left(\prod_{i=1}^{\infty} \frac{1}{1-x^i} \right)^2 F_2(x). \tag{1.8}$$

By (1A) $F_2(x) = \prod_{i=1}^{\infty}(1-x^{2i})(1+x^i)$, so that $F_2(x) = \prod_{i=1}^{\infty}(1-x^i)(1+x^i)^2$, and (1.8) follows. This completes the proof.

We now consider an orthogonal group $\mathrm{O}(V)$. Let B_0 be the principal 2-block of $\mathrm{O}(V)$ and $\mathscr{W}(B_0)$ the number of B_0-weights. Then $\mathscr{W}(B_0)$ is given by [3, (6D)]. If $\dim V$ is odd, then in the notation of [3, (6D)], $\mathscr{W}(B_0)$ is the number

$$f_{X-1} = \sum_{\kappa_1, \kappa_2, \kappa} f_{X-1, \kappa_1, \kappa_2, \kappa}, \tag{1.9}$$

where κ_1 and κ_2 run over all 2-cores such that $|\kappa_1|$ and $|\kappa_2|$ are odd and even, respectively, κ runs over all 2-cores and $f_{X-1,\kappa_1,\kappa_2,\kappa}$ is the number of 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of partitions λ_i such that

$$|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| = \frac{1}{4}(\dim V - |\kappa_1| - |\kappa_2| - 2|\kappa|).$$

Let $D(V)$ be the discriminant of V and σ a non-square element of $\mathbb{F}_q$. If $\dim V$ is even, then $\mathscr{W}(B_0)$ is also given by (1.9), where κ_1 and κ_2 run over all 2-cores such that $D(V) = \sigma^{|\kappa_1|}$ and $|\kappa_1|, |\kappa_2|$ are either both odd or both even, κ runs over all 2-cores and $f_{X-1,\kappa_1,\kappa_2,\kappa}$ is the number of 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of partitions λ_i such that

$$|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| = \frac{1}{4}(\dim V - |\kappa_1| - |\kappa_2| - 2|\kappa|).$$

If $\dim V = w$ and the type $\eta(V)$ of V is $\pm$, then we denote by $\mathscr{W}_w^{\pm}$ the number $\mathscr{W}(B_0)$. Thus $\mathscr{W}_w^+ = \mathscr{W}_w^-$ if w is odd. Let $g(t)$ be the generating function of $\mathscr{W}_w^+ + \mathscr{W}_w^-$, so that

$$g(t) = \left(\sum_{w=1}^{\infty} k(4, w) t^{4w} \right) F_2(t^2) F_2(t)^2.$$

By (1.4) and (1.7)

$$\begin{aligned} g(t) &= P(t^4)^4 P(t^4)^{-2} P(t^2) P(t^2)^{-4} P(t)^2 = P(t^4)^2 P(t^2)^{-3} P(t)^2 \\ &= \prod_{k=1}^{\infty} \frac{1+t^k}{(1-t^k)(1+t^{2k})^2}. \end{aligned}$$

On the other hand, if $\mathscr{U}_w^{\pm}$ is the number of unipotent conjugacy classes of $\mathrm{O}(V)$ such that $\dim V = w$ and $\eta(V) = \pm$, then by [14, (2.6.17)], the generating function of $\mathscr{U}_w^+ + \mathscr{U}_w^-$ is

$$F_+^+(t) = \prod_{k=1}^{\infty} \frac{(1+t^{2k-1})^2}{1-t^{2k}}.$$

But

$$\prod_{k=1}^{\infty}(1+t^k)^2 = \prod_{k=1}^{\infty}(1+t^{2k-1})^2(1+t^{2k})^2, \tag{1.10}$$

so it follows that $g(t) = F_+^+(t)$. In particular, if w is odd, then $\mathscr{W}(B_0) = \mathscr{W}_w^+ = \mathscr{W}_w^-$ is the number of unipotent conjugacy classes of $\mathrm{O}(V)$.

Suppose $w = 2n$ for some integer n. Then

$$F_2(t) + F_2(-t) = \sum_\kappa 2t^{|\kappa|} \quad F_2(t) - F_2(-t) = \sum_{\kappa'} 2t^{|\kappa'|},$$

where κ and κ' run over all 2-cores such that $|\kappa|$ and $|\kappa'|$ are even and odd, respectively. Thus the generating function $h(t)$ of $\mathscr{W}_{2n}^+ - \mathscr{W}_{2n}^-$ is given by

$$\left(\sum_{j=1}^{\infty} k(4, j)t^{4j}\right) F_2(t^2)\left[\left(\frac{1}{2}(F_2(t)+F_2(-t))\right)^2 - \left(\frac{1}{2}(F_2(t)-F_2(-t))\right)^2\right],$$

so that $h(t) = P(t^4)^4 F_2(t^2)F_2(t)F_2(-t)$. By (1.4) and (1.7)

$$\begin{aligned} h(t) &= P(t^4)^4P(t^4)^{-2}P(t^2)P(t^2)^{-2}P(t)P((-t)^2)^{-2}P(-t) \\ &= P(t^4)^2P(t^2)^{-3}P(t)P(-t) \\ &= \prod_{k=1}^{\infty}\frac{1}{(1-t^{4k})^2}\prod_{k=1}^{\infty}(1-t^{2k})^3\prod_{k=1}^{\infty}\frac{1}{(1-t^k)(1-(-t)^k)} \\ &= \prod_{k=1}^{\infty}\frac{1}{(1-t^{4k})^2}\prod_{k=1}^{\infty}(1-t^{2k})^3\prod_{k=1}^{\infty}\frac{1}{(1-t^{2k})^2}\prod_{k=1}^{\infty}\frac{1}{(1-t^{4k-2})} \\ &= \prod_{k=1}^{\infty}\frac{1}{(1-t^{4k})^2}\prod_{k=1}^{\infty}(1-t^{2k})\prod_{k=1}^{\infty}\frac{1}{(1-t^{4k-2})}. \end{aligned}$$

But $\prod_{k=1}^{\infty}(1-t^{4k-2})^{-1} = \prod_{k=1}^{\infty}(1+t^{2k})$ (cf. [14, p. 42]), so

$$h(t) = \prod_{k=1}^{\infty}\frac{1}{(1-t^{4k})^2}\prod_{k=1}^{\infty}(1-t^{4k}) = P(t^4).$$

By [14, (2.6.18)] $P(t^4)$ is the generating function $F_+^-(t)$ of $\mathscr{U}_{2n}^+ - \mathscr{U}_{2n}^-$. Thus $h(t) = F_+^-(t)$ and then $\mathscr{W}(B_0)$ is the number of unipotent conjugacy classes of $\mathrm{O}(V)$. So we have proved the following proposition.

Proposition (1D). *Let B_0 be the principal 2-block of $\mathrm{O}(V)$, and let $\mathscr{W}(B_0)$ be the number of B_0-weights. Then $\mathscr{W}(B_0)$ is the number $\mathscr{U}_{\mathrm{O}(V)}(1)$ of unipotent conjugacy classes of $\mathrm{O}(V)$. In particular, if $D(V)$ is the discriminant of V and σ is a non-square element of $\mathbb{F}_q$, then*

$$\mathscr{U}_{\mathrm{O}(V)}(1) = \sum_{\kappa_1, \kappa_2, \kappa} f_{X-1, \kappa_1, \kappa_2, \kappa},$$

where $\kappa_1, \kappa_2, \kappa$ run over all 2-cores such that $D(V) = \sigma^{|\kappa_1|}$ and $f_{X-1,\kappa_1,\kappa_2,\kappa}$ is the number of 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of partitions λ_i such that

$$|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| = \frac{1}{4}(\dim V - |\kappa_1| - |\kappa_2| - 2|\kappa|).$$

Now let B be a 2-block of $I(V)$ covering $\mathscr{E}_2(I_0(V), (s))$ for some semisimple $2'$-element s of $I_0(V)^*$, and let $s = \prod_\Gamma s_\Gamma$ be the primary decomposition of s in G^* in the sense of [11, p. 125]. In addition, let $m_\Gamma(s)$ be the multiplicity of Γ in s, let V_{X-1}^* be the underlying space of s_{X-1}, and let V_{X-1} be the space dual of V^*_{X-1} in the sense of [11, (3.1)]. By [3, (6E)] the number $\mathscr{W}(B)$ of B-weights is $\prod_\Gamma f_\Gamma$, where f_Γ is the number of partitions of rank $m_\Gamma(s)$ except when $\Gamma = X-1$, in which case f_{X-1} is given by (1.6) or (1.9) with $\dim V$ replaced by $\dim V_{X-1}$ according as V is symplectic or orthogonal. Thus f_{X-1} is the number of unipotent conjugacy classes of $I(V_{X-1})$ by (1C) and (1D). By [11, (1.13)]

$$C_{I_0(V)^*}(s)^* \simeq \prod_{\Gamma \neq X-1} \mathrm{GL}(m_\Gamma(s), \varepsilon_\Gamma q^{\delta_\Gamma}) \times I_0(V_{X-1}),$$

where ε_Γ and δ_Γ are defined by [11, (1.8) and (1.9)] and $\mathrm{GL}(m, -q^\delta) = \mathrm{U}(m, q^\delta)$ for all $\delta \geq 1$. Let s^* be a dual of s in $I_0(V)$ defined by [3, (4A)] with the primary decomposition $\prod_\Gamma s^*_\Gamma$. By [3, (4.1)]

$$C_{I_0(V)}(s^*) \simeq C_{I_0(V)^*}(s)^*$$

and by definition, $m_\Gamma(s^*) = m_\Gamma(s)$ for $\Gamma \neq X-1$ and V_{X-1} is the underlying space of s^*_{X-1}, therefore

$$C_{I(V)}(s^*) \simeq \prod_{\Gamma \neq X-1} \mathrm{GL}(m_\Gamma(s^*), \varepsilon_\Gamma q^{\delta_\Gamma}) \times I(V_{X-1}).$$

But the number of unipotent conjugacy classes of $\mathrm{GL}(m, \varepsilon q^\delta)$ for any sign $\varepsilon = \pm$ and $\delta \geq 1$ is the number of partitions of rank m, so $\mathscr{W}(B)$ is the number of unipotent conjugacy classes of $C_{I(V)}(s^*)$. Thus we have proved the following.

Proposition (1E). *Let B be a 2-block of $I(V)$ with a semisimple label s^* for some semisimple $2'$-element s^* of $I_0(V)$. Then the number $\mathscr{W}(B)$ of B-weights is the number of unipotent conjugacy classes $\mathscr{U}_{I(V)}(s^*)$ of $C_{I(V)}(s^*)$. In particular, if $\prod_\Gamma s^*_\Gamma$ is the primary decomposition of s^* in $I(V)$ in the sense of* [11, p. 125], *then*

$$\mathscr{U}_{I(V)}(s^*) = \prod_\Gamma f_\Gamma,$$

where f_Γ is the number of partitions of multiplicity $m_\Gamma(s^*)$ of the elementary divisor Γ in s^* except when $\Gamma = X-1$, in which case f_{X-1} is given by (1.6) *or* (1.9) *with V replaced by the underlying space of s^*_{X-1} according as V is symplectic or orthogonal.*

Remark (1F). As a corollary of (1E), we can get an affirmative answer for Alperin's weight conjecture for $I(V)$. Indeed, if $\mathscr{W}(I(V))$ is the number of weights of $I(V)$, then by (1E),

$$\mathscr{W}(I(V)) = \sum_{s^*} \mathscr{U}_{I(V)}(s^*), \tag{1.11}$$

where s^* runs over all semisimple $2'$-elements of $I(V)$. Now the right-hand side of (1.11) is the number of conjugacy classes of 2-regular elements in G and it is the number of irreducible Brauer characters $l(I(V))$ of $I(V)$ by a result of Brauer. Thus $\mathscr{W}(I(V)) = l(I(V))$ and the remark follows.

2. The number of irreducible Brauer characters

The notation and terminology of §1 are continued in this section. The number of irreducible Brauer characters in a 2-block of a symplectic or special orthogonal group will be given and the weight conjecture of Alperin will be confirmed block by block for a symplectic or odd-dimensional orthogonal group.

The proof of the following proposition was pointed out by the referee of [3].

Proposition (2A). *Let q be a power of an odd prime, $G = I_0(V)$, B a 2-block of G, and $l(B)$ the number of irreducible Brauer characters in B. If $B = \mathscr{E}_2(G, (s))$ for some semisimple $2'$-element s of the dual group $G^* = I_0(V)^*$, then $l(B)$ is the number of unipotent conjugacy classes of $C_{G^*}(s)^*$.*

Proof. Let t^* be a semisimple $2'$-element of G, and let t be its dual given by [3, (4A)], so that $C_G(t) \simeq C_{G^*}(t)^*$. Let $\mathscr{U}(t)^*$ be the number of unipotent conjugacy classes of $C_G(t^*)$. If $\dim V \leq 2$, then it is trivial to check that $l(B) = \mathscr{U}(s^*)$. Suppose $s \neq 1$. Then $C_{G^*}(s)$ is a proper regular subgroup of G^* and $C_G(s^*)$ is its dual group. By Broué [5, Theorem 2.3] there is a perfect isometry between B and $\mathscr{E}_2(C_G(s^*), (1))$. It follows that $l(B)$ is the number of irreducible Brauer characters of $\mathscr{E}_2(C_G(s^*), (1))$. Let $\prod_\Gamma s_\Gamma^*$ be the primary decomposition of s^* in G and V_{X-1} the underlying space of s_{X-1}^*. Then

$$C_G(s^*) \simeq \left(\prod_{\Gamma \neq X-1} \mathrm{GL}(m_\Gamma(s), \varepsilon_\Gamma q^{\delta_\Gamma}) \right) \times I_0(V_{X-1})$$

and $\dim V_{X-1} < \dim V$, so by induction $l(\mathscr{E}_2(I_0(V_{X-1})), (1))$ is the number of unipotent conjugacy classes of $I_0(V_{X-1})$. By [10, §8] $l(\mathrm{GL}(m_\Gamma(s), e_\Gamma q^{\delta_\Gamma}))$ is the number of partitions of rank $m_\Gamma(s)$. Thus $l(B) = \mathscr{U}(s^*)$. Now let $s = 1$. The number of irreducible Brauer characters $l(G)$ of G is

$$l(B_0) + \sum_{t^* \neq 1} \mathscr{U}(t^*), \tag{2.1}$$

where t^* runs over all representatives for the semisimple conjugacy $2'$-classes of G with $t^* \neq 1$. By a result of Brauer $l(G)$ is the number of conjugacy $2'$-classes in G, so that

$$l(G) = \mathscr{U}(1) + \sum_{t^* \neq 1} \mathscr{U}(t^*).$$

It follows that $l(B_0) = \mathscr{U}(1)$ and this proves (2A).

Proposition (2B). *Let V be a symplectic or odd-dimensional orthogonal space and B a 2-block of $I(V)$ such that B covers a 2-block $B' = \mathscr{E}_2(I_0(V), (s))$ of $I_0(V)$, where s is a semisimple $2'$-element of the dual group $I_0(V)^*$ of $I_0(V)$. Let s^* be a dual of s in $I_0(V)$ given by* [3, (4A)], *and let $\mathscr{U}_{I(V)}(s^*)$ be the number of unipotent conjugacy classes of $C_{I(V)}(s^*)$. In addition, let $\mathscr{W}(B)$ be the number of B-weights. If $l(B)$ is the number of irreducible Brauer characters in B, then $l(B) = \mathscr{W}(B) = \mathscr{U}_{I(V)}(s^*)$. In particular, Alperin's weight conjecture holds for B block by block.*

Proof. Let $\mathscr{U}_{I_0(V)}(s^*)$ be the number of unipotent conjugacy classes of $C_{I_0(V)}(s^*)$. If V is symplectic, then $I(V) = I_0(V)$ and $C_{I(V)}(s^*) \simeq C_{I(V)^*}(s)^*$. By (2A) and (1E) $l(B) = \mathscr{U}_{I(V)}(s^*) = \mathscr{W}(B)$. If V is odd-dimensional orthogonal, then $I(V) = \langle -1_V \rangle \times G$ and $C_{I(V)}(s^*) = \langle -1_V \rangle \times C_{I_0(V)}(s^*)$, where 1_V is the identity of $I(V)$. Thus $\mathscr{U}_{I(V)}(s^*) = \mathscr{U}_{I_0(V)}(s^*)$ and $l(B) = l(B')$, so that (2B) follows from (2A) and (1E).

Remark (2C). In the notation of (2B), suppose V is even-dimensional orthogonal. If the multiplicity $m_{X-1}(s^*)$ of $X-1$ in s^* is zero, then $\mathscr{U}_{I(V)}(s^*) = l(B)$ and so Alperin's weight conjecture has an affirmative answer for B. Indeed, in this case $C_{I(V)}(s^*) = C_{I_0(V)}(s^*)$, so that $\mathscr{U}_{I(V)}(s^*) = \mathscr{U}_{I_0(V)}(s^*) = l(B')$. Let τ be an involution of $I(V^*)$ with determinant -1, where V^* is the underlying space of $I_0(V)^*$. Then $\mathscr{E}_2(I_0(V), (s^\tau))$ is a block B'' of $I_0(V)$ and B covers B'' by [3, (5B)]. Since s is not conjugate with s^τ in $I_0(V)^*$, it follows that $B' \neq B''$, so that $B'^\tau = B''$ and $\tau \notin N(B')$, where $N(B')$ is the stabilizer of B' in $I(V)$. Thus $N(B') = I_0(V)$ and $l(B) = l(B')$ by a result of Fong and Reynolds [9, Theorem V.2.5]. It follows that $\mathscr{U}_{I(V)}(s^*) = l(B)$ and the remark follows.

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