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A COUNTEREXAMPLE TO A COMPACT EMBEDDING THEOREM FOR FUNCTIONS WITH VALUES IN A HILBERT SPACE

STANISLAW MIGORSKI

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ABSTRACT. A counterexample to a compactness embedding result of Nagy is provided.

Let V and H be real separable Hilbert spaces with V densely and continuously embedded in H. Identifying H with its dual we write $V \subset H \subset V'$ algebraically and topologically, where V' is the dual space to V.

Given $T > 0$, let $\mathscr{V} = L^2(0, T; V)$, $\mathscr{H} = L^2(0, T; H)$ and $\mathscr{V}' = L^2(0, T; V')$ denote the spaces of the square summable functions defined on the interval $(0, T)$ with values in V, H and V', respectively. We define $\mathscr{W} = \{v \in \mathscr{V} : v' \in \mathscr{V}'\}$, which with the usual norm $\|u\|_{\mathscr{W}} = (\|u\|^2_{\mathscr{V}} + \|u'\|^2_{\mathscr{V}'})^{1/2}$ is a Hilbert space (cf. [7], Theorem 25.4). The following remarks about the space $\mathscr{W}$ are in order. In the definition the derivative $u' = \frac{du}{dt}$ is understood in the weak sense. Nevertheless, when we view u as a V'-valued function, we know (see [8], Proposition 23.23) that it is absolutely continuous, so its derivative exists in the strong sense almost everywhere on $(0, T)$. It is known (see, e.g., [7], Theorem 25.5) that the embedding

$$\mathscr{W} \subset C(0, T; H) \tag{1}$$

is continuous. Here $C(0, T; H)$ denotes the space of continuous functions from $[0, T]$ into H, endowed with the supremum norm. Therefore every function in $\mathscr{W}$ can be, after modification on the set of measure zero, considered as an element of $C(0, T; H)$. We also know that $\mathscr{W} \subset \mathscr{H}$ compactly (see [8]). The following is the main result of Nagy (see [3], Theorem 2).

Theorem. *Let V and H be infinite-dimensional separable Hilbert spaces such that the embedding $V \subset H$ is dense, continuous and compact. Then the embedding (1) is also compact.*

The aim of this paper is to exhibit a simple example which shows that the embedding (1) cannot be compact, i.e., the above theorem is not true. It should also be noted here that this theorem was exploited in several recent papers in

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connection with the study of properties of solutions to parabolic differential equations (see, for example, [1], [2], [4], [5], [6]).

1. An example

Let us consider the following separable Hilbert spaces:

$$H = l^2 = \left\{ x = (x^k)_{k=1}^{\infty} \colon \sum_{k=1}^{\infty} |x^k|^2 < +\infty \right\},$$

$$V = \left\{ x = (x^k)_{k=1}^{\infty} \colon \sum_{k=1}^{\infty} k|x^k|^2 < +\infty \right\},$$

$$V' = \left\{ x = (x^k)_{k=1}^{\infty} \colon \sum_{k=1}^{\infty} \frac{1}{k}|x^k|^2 < +\infty \right\}$$

furnished with the standard norms

$$\|x\|_H = \left(\sum_{k=1}^{\infty} |x^k|^2 \right)^{\frac{1}{2}}, \quad \|x\|_V = \left(\sum_{k=1}^{\infty} k|x^k|^2 \right)^{\frac{1}{2}}, \quad \|x\|_{V'} = \left(\sum_{k=1}^{\infty} \frac{1}{k}|x^k|^2 \right)^{\frac{1}{2}}.$$

The above three spaces can be identified with the Gelfand triple $H_0^1(I) \subset L^2(I) \subset H^{-1}(I)$ on an open bounded interval $I \subset \mathbf{R}$. Hence and from Rellich's theorem, we know that $V \subset H \subset V'$ with compact embeddings.

The compactness of the embedding (1) means that every sequence bounded in $\mathscr{W}$ possesses a subsequence converging in $C(0, T; H)$. In order to show that (1) is not compact, it is enough to find a sequence $(u_n)_{n\in\mathbf{N}}$ of functions such that

(2)
$\{u_n\}$ is bounded in $\mathscr{W}$, i.e., u_n is bounded in $\mathscr{V}$ and u_n' is bounded in $\mathscr{V}'$;

(3) $\exists C > 0 \colon \forall m, n \in \mathbf{N}, \ m \neq n, \quad \|u_n - u_m\|_{C(0,T;H)} \geq C.$

We define the sequence $u_n(t) = (u_n^k(t))_{k=1}^{\infty}$ by

$$u_n^k(t) = 0 \quad \text{for every } k \neq n,$$

$$u_n^n(t) = \begin{cases} 1 - nt, & \text{if } 0 \leq t \leq \frac{1}{n}, \\ 0, & \text{if } \frac{1}{n} \leq t \leq T. \end{cases}$$

Then we have

$$\|u_n\|_{\mathscr{V}}^2 = \int_0^T \|u_n(t)\|_V^2 \, dt = \int_0^T \sum_{k=1}^{\infty} k|u_n^k(t)|^2 \, dt$$

$$= \int_0^T n|u_n^n(t)|^2 \, dt = \int_0^{\frac{1}{n}} (1 - nt)^2 \, dt = \frac{1}{3}$$

and

$$\|u_n'\|_{\mathscr{V}'}^2 = \int_0^T \|u_n'(t)\|_{V'}^2 \, dt = \int_0^T \sum_{k=1}^{\infty} \frac{1}{k} |(u_n^k(t))'|^2 \, dt$$

$$= \frac{1}{n} \int_0^T |(u_n^n(t))'|^2 \, dt = \frac{1}{n} \int_0^{\frac{1}{n}} n^2 \, dt = 1.$$

Hence, it follows that the sequence (u_n) satisfies condition (2). Now let $m, n \in \mathbf{N}$, $m < n$. We easily obtain

$$\|u_n(0) - u_m(0)\|_H^2 = \sum_{k=1}^{\infty} |u_n^k(0) - u_m^k(0)|^2 = |u_m^m(0)|^2 + |u_n^n(0)|^2 = 2,$$

which immediately implies

$$\|u_n - u_m\|_{C(0,T;H)} \geq \|u_n(0) - u_m(0)\|_H = \sqrt{2}.$$

This shows that (3) holds. Consequently, the embedding (1) is not compact.

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Institute for Information Sciences, Jagellonian University, ul. Nawojki 11, 30072 Cracow, Poland

E-mail address: migorski@softlab.ii.uj.edu.pl