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FINDING DISCONTINUITIES FROM TOMOGRAPHIC DATA

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ABSTRACT. A method for derivation of inversion formulas for the Radon transform of $f(x)$ is given. This method allows one to construct functions $\psi(x)$ which are easy to compute and which have the same wave fronts as $f(x)$. This, in turn, allows one to calculate the singular support S of $f(x)$ by computing $\psi(x)$ given tomographic data. A simple geometrical relation between the singular supports of f and $\hat{f}$, its Radon transform, is formulated (the duality law).

I. INTRODUCTION

Let $f(x)$ be a piecewise-smooth function of the form

$$f(x) = \sum_{j=1}^{J} \chi_{D_j}(x)\Phi(x),$$

where $\Phi(x) \in C_0^2(\mathbb{R}^n)$, $\chi_{D_j} = \begin{cases} 1, & x \in D_j, \\ 0, & x \notin D_j, \end{cases}$ D_j is a connected bounded domain in $\mathbb{R}^n$, $S_j := \partial D_j$ is its boundary, which is a union of finitely many smooth hypersurfaces, $D_j \cap D_i = \varnothing$ for $j \neq i$.

Let $\hat{f}(\alpha, p) := \int_{\ell_{\alpha p}} f(x)\,ds$ be the Radon transform of $f(x)$, $\ell_{\alpha p} = \{x : \alpha \cdot x = p\}$, $\alpha \cdot x$ is the dot product in $\mathbb{R}^n$, $p \in \mathbb{R}^1$, $\alpha \in S^{n-1}$, S^{n-1} is the unit sphere in $\mathbb{R}^n$, ds is the Euclidean measure on the plane $\ell_{\alpha p}$. The function $\hat{f}(\alpha, p)$ is the tomographic data (if $n = 2$).

The objectives of this paper are:

(1) to give a simple derivation of the inversion formulas which allow one to calculate $f(x)$ given $\hat{f}(\alpha, p)$; this derivation uses a special distribution and the derivation makes it clear when one gets *local* inversion formulas; although there are many derivations of the inversion formulas for the Radon transform (see [1]-[2]), the one given here is based on a new idea and is shorter and more direct than the known ones;

(2) to give a derivation of a local inversion formula which yields a function ψ with the same wave front $WF\psi$ as $WFf(x)$, but is easy to compute;

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(3) to apply the result for finding the singular support of $f(x)$ given $\hat{f}(\alpha, p)$; in particular, to justify local tomography [3] as a particular (and the simplest) case of our formula (15): the local tomography in the sense [3] one gets from (15) if $g(\alpha) = 1$. The function g in (15) is an arbitrary function which satisfies conditions (13) (see formula (13) below). The freedom in the choice of this function can be used for improving the noise stability of the method for finding discontinuity surfaces of $f(x)$ which is based on formula (15) (see [17]). By the singular support of $f(x)$, $\operatorname{singsupp} f$, we mean the complement to the largest open set U of points x such that the restriction of f to a neighborhood of x is an infinitely differentiable function; and

(4) to formulate the duality law, a simple geometrical relation between singular supports of f and $\hat{f}$.

In section II the inversion formula is derived. In section III a family of functions ψ is constructed, $WF(\psi) = WF(f)$, and in section IV applications are discussed. In section V an elementary derivation of a result concerning wave fronts is given, and in section VI the duality law is formulated. This paper is based on the previous work [4–10], but can be read independently. Related works are [15], [16], [18–22]. In the monograph [22] the new methods for finding discontinuities from tomographic data are developed systematically.

II. Inversion formula

It is well known and easy to check that

$$(1) \quad \int_{-\infty}^{\infty} \hat{f}(\alpha, p) \exp(i\lambda p) dp = \tilde{f}(\xi) := \mathscr{F} f, \quad \lambda \in \mathbb{R}_+, \quad |\xi| := \lambda, \quad \alpha := \frac{\xi}{|\xi|},$$

$$(2) \qquad \tilde{f}(\lambda\alpha) := \int f(x) \exp(i\lambda\alpha \cdot x) dx, \quad \int := \int_{\mathbb{R}^n},$$

$$(3) \qquad f(x) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\alpha \int_0^{\infty} d\lambda \lambda^{n-1} \tilde{f}(\lambda\alpha) \exp(-i\lambda\alpha \cdot x),$$

$$(4) \qquad \hat{f}(-\alpha, -p) = \hat{f}(\alpha, p),$$

$$(5) \quad \int_0^{\infty} d\lambda \lambda^n \exp(i\lambda\sigma) = i^{n+1} n! \sigma^{-n-1} + (-i)^n \pi \delta^{(n)}(\sigma) = i^{n+1} n! (\sigma + i0)^{-n-1}.$$

Here $\delta(\sigma)$ is the delta-function. Formula (5) can be found in [12] and can be easily derived by differentiation with respect to σ (in the sense of distributions) of the elementary formula $\int_0^\infty \exp(i\lambda\sigma) d\lambda = i(\sigma + i0)^{-1} = i\sigma^{-1} + \pi\delta(\sigma)$. To derive the inversion formula (following [4]), substitute (1) in (3), change the order of integration (which is possible in the sense of distributions), and get:

$$(6) \quad \begin{aligned} f(x) &= \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} dp \hat{f}(\alpha, p) \int_0^{\infty} d\lambda \lambda^{n-1} \exp\{i\lambda(p - \alpha \cdot x)\} \\ &= \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\alpha \int_{-\infty}^{\infty} dp \hat{f}(\alpha, p) \cdot \\ &\quad \cdot \left[i^n (n-1)! (p - \alpha \cdot x)^{-n} + (-i)^{n-1} \pi \delta^{(n-1)}(p - \alpha \cdot x) \right]. \end{aligned}$$

Formula (6) yields the desired inversion formula. For instance, consider the case $n = 3$. Using (4) one concludes

$$\int_{S^2} d\alpha \int_{-\infty}^{\infty} dp\hat{f}(\alpha, p)(p - \alpha \cdot x)^{-3} = 0, \tag{7}$$

and

$$\frac{\pi(-i)^2}{(2\pi)^3} \int_{S^2} d\alpha \int_{-\infty}^{\infty} dp\hat{f}(\alpha, p)\delta^{(2)}(p - \alpha \cdot x) = \frac{-1}{8\pi^2} \int_{S^2} d\alpha \frac{\partial^2 \hat{f}(\alpha, p)}{\partial p^2}|_{p=\alpha \cdot x}. \tag{8}$$

Thus

$$f(x) = -\frac{1}{8\pi^2} \int_{S^2} d\alpha \hat{f}_{pp}(\alpha, \alpha \cdot x), \quad \hat{f}_{pp} := \frac{\partial^2 \hat{f}}{\partial p^2}. \tag{9}$$

This is the well-known inversion formula for $\hat{f}(\alpha, p)$ in $\mathbb{R}^3$. Consider now the case $n = 2$. Then (4) yields

$$\int_{S^1} d\alpha \int_{-\infty}^{\infty} dp\hat{f}(\alpha, p)\delta'(p - \alpha \cdot x) = 0 \tag{10}$$

and

$$f(x) = -\frac{1}{(2\pi)^2} \int_{S^1} d\alpha \int_{-\infty}^{\infty} dp \frac{\hat{f}(\alpha, p)}{(p - \alpha \cdot x)^2}, \tag{11}$$

where the distribution $\frac{1}{(p-\alpha\cdot x)^2}$ is defined in [12], formula (7), Ch. 1, §3:

$$\int_{-\infty}^{\infty} \frac{\phi dx}{x^2} = \int_0^{\infty} \frac{\phi(x) + \phi(-x) - 2\phi(0)}{x^2} dx, \quad \phi \in C_0^2.$$

Formula (11) is a known inversion formula in $\mathbb{R}^2$. If one integrates by parts in (11), one gets:

$$f(x) = -\frac{1}{(2\pi)^2} \int_{S^1} d\alpha \int_{-\infty}^{\infty} \hat{f}_p(\alpha, p) \frac{dp}{p - \alpha \cdot x}, \quad \hat{f}_p(\alpha, p) := \frac{\partial \hat{f}(\alpha, p)}{\partial p} \tag{12}$$

where the integral inside is the principal value integral. If $n = 2m$, $m = 1, 2, \dots$, the functional $\delta^{(2m-1)}(p-\alpha\cdot x)$ annihilates $\hat{f}(\alpha, p)$ due to condition (4). If $n = 2m+1$, $m = 1, 2, \dots$, the functional $(p-\alpha\cdot x)^{-(2m+1)}$ annihilates $\hat{f}(\alpha, p)$ due to condition (4). Thus, for n even or odd, formula (6) yields an inversion formula for $\hat{f}(\alpha, p)$.

III. Local tomography formulas

Multiply (1) by $\frac{1}{(2\pi)^n}\lambda^{n-1+\gamma} g(\alpha) \exp(-i\lambda\alpha \cdot x)$, where $\gamma > 0$ is an integer,

$$g(\alpha) \in C^\infty(S^{n-1}), \quad g(\alpha) = g(-\alpha), \quad 0 < c_1 \le g(\alpha) \le c_2 < \infty, \tag{13}$$

and integrate over $(0, \infty)$ with respect to (wrt) λ and over $S^{n-1} wrt$ α to get:
(14)

$$\begin{aligned}\psi(x) :&= \frac{1}{(2\pi)^n}\int_{S^{n-1}} d\alpha \int_0^\infty d\lambda \lambda^{n-1+\gamma}\exp(-i\lambda\alpha\cdot x)\tilde{f}(\lambda\alpha)g(\alpha)\\ &= \frac{1}{(2\pi)^n}\int_{S^{n-1}} d\alpha g(\alpha)\int_{-\infty}^\infty dp\hat{f}(\alpha, p)\int_0^\infty d\lambda\lambda^{n-1+\gamma}\exp\{i\lambda(p-\alpha\cdot x)\}\\ &= \frac{1}{(2\pi)^n}\int_{S^{n-1}} d\alpha g(\alpha)\int_{-\infty}^\infty dp\hat{f}(\alpha, p)\\ &\qquad \left[i^{n+\gamma}(\gamma+n)!(p-\alpha\cdot x)^{-n-\gamma} + (-i)^{n-1+\gamma}\pi\delta^{(n-1+\gamma)}(p-\alpha\cdot x)\right].\end{aligned}$$

The basic idea of our derivation is very simple: if $n+\gamma$ is an odd integer, then the distribution $(p-\alpha\cdot x)^{-n-\gamma}$ annihilates $\hat{f}(\alpha, p)$ and formula (14) *yields a "local tomography" inversion formula:*

$$\psi(x) = \frac{\pi i^{n-1+\gamma}}{(2\pi)^n}\int_{S^{n-1}} d\alpha g(\alpha)\frac{\partial^{n-1+\gamma}\hat{f}(\alpha, p)}{\partial p^{n-1+\gamma}}\Big|_{p=\alpha\cdot x}. \tag{15}$$

This formula is our basic result, which gives the basis for local tomography. Take, for example, $n=2$, $\gamma=1$. Then (15) yields:

$$\psi(x) = \frac{-1}{4\pi}\int_{S^1} d\alpha g(\alpha)\hat{f}_{pp}(\alpha, \alpha\cdot x), \quad n=2. \tag{16}$$

By definition (14), $\psi(x) = \mathcal{F}^{-1}\{\lambda^\gamma g(\alpha)\tilde{f}(\lambda\alpha)\}$. We could take any odd integer γ if $n=2$. We have chosen $\gamma=1$, the smallest odd positive integer, because this choice minimizes the order of the derivative in formula (16). There is a trade-off between increasing γ, which enhances the sharpness of the discontinuity surfaces, and at the same time increases the degree of ill-posedness of the problem, since the larger γ is, the higher order derivative in formula (15) will appear.

We will prove that $WF(f) = WF(\psi)$. This implies that singsupp ψ = singsupp f.

Lemma 1. *If* (13) *holds, then $WF(f) = WF(\psi)$. In particular,* singsupp f = singsupp ψ.

Proof. We have $\mathcal{F}\psi = \lambda^\gamma g(\alpha)\mathcal{F}f$. Thus $f \mapsto \psi$ is a pseudodifferential operator (pdo) with the symbol $\lambda^\gamma g(\alpha)$, $\alpha \in S^{n-1}$. If (13) holds, then the operator $f \to \mathcal{F}^{-1}\{g(\alpha)\mathcal{F}f\} := Bf$ is an elliptic pdo of order zero (modulo a smoothing operator), so $WF(f) = WF(Bf)$, where $WF(f)$ is the wave front of $f(x)$ [13, p. 254]. The operator $f \to \mathcal{F}^{-1}(\lambda^\gamma\mathcal{F}f) := Af$ can be written as a composition of two operators each of which preserves $WF(f)$. This argument is valid for any positive integer n and any integer γ. However, the operator A will be local only if $n+\gamma$ is odd. For example, if $n=2$ and $\gamma=1$, then $Af = (-\Delta)Tf$, $Tf :=$ const $\int_{\mathbb{R}^2}\frac{f(y)dy}{|x-y|}$, where Δ is the Laplacian. In general, $Af = (-\Delta)^m Tf$, where m is an integer. The elliptic operator $-\Delta$ preserves the wave front: $WF(-\Delta f) = WF(f)$. The operator T preserves $WF(f)$. Indeed, the symbol of T is $p(\xi) = c|\xi|^{-1}$, $c \neq 0$ is a constant; write $p(\xi) = p_1(\xi) + p_2(\xi)$, $p_2 := p - p_1$, $p_1(\xi) := G(|\xi|)$, where $G(\lambda) = c\lambda^{-1}$ if $\lambda > 1$, and $G(\lambda) > c_1 > 0$ if $0 \le \lambda \le 1$, $G(\lambda) \in C^\infty([0, \infty))$. Therefore the function $p_1(\xi)$ is the symbol of an elliptic pdo T_1 which preserves the wave front sets [13]. The function $p_2(\xi)$ has compact support, is integrable, and therefore it is the symbol of an infinitely smoothing pdo. Thus $WF(Tf) = WF(T_1 f) = WF(f)$ as claimed. Since $\operatorname{singsupp} f$ is the natural projection of $WF(f)$ onto $\mathbb{R}^n$, the last conclusion of the lemma follows. $\square$

In section V an elementary proof of the relation $WF(f) = WF(Af)$ is given.

IV. Applications

Formula (15) yields a function whose singular support is the same as that of $f(x)$ (because $WF\psi(x) = WFf(x)$). Function (15) can be calculated from the data $\hat{f}(\alpha, p)$ by means of local (with respect to p) operators: differentiation with respect to p and weighted averaging with respect to α. In this sense one calls (15) a "local tomography" formula. In order to find the singular support of $f(x)$, given $\hat{f}(\alpha, p)$, one calculates $\psi(x)$ by formula (15), and the $\operatorname{singsupp} \psi = \operatorname{singsupp} f$.

One can use the choice of $g(\alpha)$ to optimize the formula in some sense (for example, to improve the resolution ability of the method for finding singularities of $f(x)$, based on formula (15)).

V. An elementary proof of the relation $WF(f) = WF(Af)$

The purpose of this section is to derive the above relation without using the results from the theory of pdo. The definition of the wave front of f, $WF(f)$, can be found in many books, e.g. in [13]. For convenience of the reader, we recall this definition. Let D be a domain in $\mathbb{R}^n$, x, ξ are vectors in $\mathbb{R}^n$, $|\xi| > 0$, U is a (small) neighborhood of the point x, $\epsilon > 0$ is sufficiently small, $x^0 := x/|x|$. The $WF(f)$ is a subset in $D \times (\mathbb{R}^n \setminus 0)$ and we now define the complement to $WF(f)$. We say that the point (x, ξ_0) does not belong to $WF(f)$ if there exists a function $\phi \in C_0^\infty(U)$, $\phi(x) = 1$ such that $|v(\xi)| < c_N(1+|\xi|^2)^{-N}$ for all $N > 0$ and ξ, and $|\xi_0^0 - \xi^0| < \epsilon$, where $v(\xi) := \mathscr{F}(\phi f)$.

Let $f \in L^2(B_a)$, $(x_0, \xi_0) \notin WF(f)$, $-n < \gamma$. Let us prove that $(x_0, \xi_0) \notin WF(Af)$. The converse is proved similarly. Let K be a cone with vertex at the origin and $\phi \in C_0^\infty(U_{x_0})$ be a function such that $|\mathscr{F}(\phi f)| < c_N(1+|\xi|^N)^{-1}$ $(*)$ for all $N > 0$ and $\xi \in K$, $\phi = 1$ in U, a neighborhood of x_0, $U \subset U_{x_0}$. By c_N and by c we denote below various constants. Let $K_1 \subset K$ be a cone with vertex at the origin, the same axis of symmetry as that of K and half of the opening of K, $\eta \in C_0^\infty(U_{x_0})$, $\eta = 1$ in $U_1 \subset U$, $x_0 \in U_1$. We want to prove that $|\mathscr{F}(\eta A f)| \le c_N(1+|\xi|^N)^{-1}$ for all $N > 0$, $\xi \in K_1$ $(**)$. Since $Af = A[\phi f] + A[(1-\phi)f]$ and $A[(1-\phi)f] := h \in C^\infty(U_1)$, it is sufficient to prove $(**)$ with ϕf in place of f. One has

$$q(\xi) := \mathscr{F}[\eta A(\phi f)] = \int \zeta(\xi - \xi')|\xi'|^\gamma \psi(\xi')d\xi', \qquad \int := \int_{\mathbb{R}^n},$$

where $\psi(\xi) := \mathscr{F}(\phi f)$ satisfies $(*)$ and $\zeta(\xi) := \mathscr{F}\eta$. Let $\xi \in K_1$, $\xi' \in \mathbb{R}^n \setminus K$. Then $c|\xi| < |\xi - \xi'|$, $c|\xi'| < |\xi - \xi'|$, $c > 0$. Note that $|\psi(\xi)| < c$ $\forall \xi \in \mathbb{R}^n$. Using the fact that $|\zeta(\xi)|$ decays faster than any power of $|\xi|$, one gets:

$$
\begin{aligned}
|q(\xi)| &\le c_N \int_K |\xi'|^\gamma (1+|\xi-\xi'|^N)^{-1}(1+|\xi'|^{N+n+1+\gamma})^{-1}\,d\xi' \\
&\quad + c_N \int_{\mathbb{R}^n\setminus K} |\xi'|^\gamma |\psi(\xi')| (1+|\xi-\xi'|^{N+n+1+\gamma})^{-1}\,d\xi' \\
&\le c_N(1+|\xi|^N)^{-1} + c_N \int_{\mathbb{R}^n\setminus K} (1+|\xi'|^{n+1})^{-1}(1+|\xi-\xi'|^N)^{-1}\,d\xi' \\
&\le c_N(1+|\xi|^N)^{-1}.
\end{aligned}
$$

The proof is complete. It is well known that the relation $WF(f) = WF(Af)$ implies the relation $\operatorname{singsupp} f = \operatorname{singsupp}(Af)$, which is of interest for us.

VI. Duality law

Let us formulate a duality law (see also [4], [14]) which describes the singular support sets S of $f(x)$ and $\hat{S}$ of $\hat{f}$. Let $x_n = g(x')$, $x' := (x_1, \dots, x_{n-1})$ be the local equation of S, and $q = h(\beta)$, $\beta \in \mathbb{R}^{n-1}$ be the local equation of $\hat{S}$ (see [4]–[6]). It is proved in [4]–[6] that $g = Lh$ and $h = Lg$, where L is the Legendre transform. Let $0 = \beta \cdot x' - q - x_n$ $(*)$ be the equation of the plane tangent to S. If $q = h(\beta)$, where $h(\beta)$ is so chosen, that $(*)$ is a family of tangent planes to S, then the envelope of this family is S and β is the parameter of the family. Consider $(*)$ as a family of the planes tangent to $\hat{S}$, x' being the parameter, $x_n = g(x')$. This is possible because: (i) $\beta \cdot x' - q - x_n$ is a linear function of β and q, (ii) the values β and q in equation $(*)$ belong to $\hat{S}$ by the assumption, and (iii) $(x', -1)$ is the normal to $\hat{S}$ at the point (β, q), where β, q are the parameters from equation $(*)$. Only the last claim needs a proof. The normal in (iii) is $(\nabla h, -1)$, and $\nabla h = x'$ by the property of the envelope of the family of the tangent to S planes, which is given by the planes $(*)$ with $q = h(\beta)$. Thus if $(x', g(x'))$ is a point on S at which the plane $(*)$ is tangent to S, then (iii) holds. Using the classical methods for calculating the envelopes, one gets the relations $g = Lh$, $h = Lg$ and concludes that both S and and $\hat{S}$ are the envelopes of the family of planes $(*)$, β is the parameter when S is the envelope, and x' is the parameter when $\hat{S}$ is the envelope, $x_n = g(x')$ is the equation of S and $q = h(\beta)$ is the equation of $\hat{S}$ in the local coordinates. This is the duality law we wanted to formulate (see also [4], [14]). A more traditional form of the equation of the tangent plane to S is $\alpha \cdot x = p$ $(**)$. If one takes $p = p(\alpha)$ in $(**)$ so that $(**)$ becomes a family of the planes, tangent to S, parametrized by n-dimensional vector α, then the envelope of this family is S, and the equation $p = p(\alpha)$ is the equation of $\hat{S}$. If S is a variety in $\mathbb{R}^n$, then the set of points (α, p) such that the planes $\alpha \cdot x = p$ are tangent to S is called the dual variety to S. The results of this section can be summarized as follows.

Proposition. *The discontinuity surfaces S of $f(x)$ and the discontinuity surfaces $\hat{S}$ of $\hat{f}$ are dual varieties whose local equations are related via the Legendre transform, which is an involution.*

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