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ON SOLUTIONS OF ELLIPTIC EQUATIONS THAT DECAY RAPIDLY ON PATHS

D. H. ARMITAGE

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ABSTRACT. Let $P(D)$ be an elliptic differential operator on $\mathbb{R}^n$ with constant coefficients. It is known that if u is a solution of $P(D)u = 0$ on an unbounded domain and if u decays uniformly and sufficiently rapidly, then $u = 0$. In this note it is shown that the same conclusion holds if u decays rapidly, but not a priori uniformly, on a sufficiently large set of unbounded paths.

Throughout this note Ω is an unbounded domain in $\mathbb{R}^n$, where $n \geq 2$, and $P(D) = \sum_{|\alpha| \leq d} a_\alpha D^\alpha$ is an elliptic linear differential operator on $\mathbb{R}^n$ with constant complex coefficients. In response to a problem proposed for the harmonic case ($P(D) = \Delta$, the Laplacian operator) at a Durham Conference in 1983 [3, Problem 3.27], Armitage, Bagby and Gauthier [1] gave two proofs of the following result.

Theorem A. *There exists a continuous function* $\varepsilon\colon [0, +\infty) \to (0, 1]$ *with the following property. If* u *is a solution of* $P(D)u = 0$ *on* Ω *such that* $|u(x)| \leq \varepsilon(\|x\|)$ *for all* $x \in \Omega$, *then* $u = 0$.

Some theorems for special domains Ω in the harmonic and holomorphic ($n = 2$ and $P(D) = \overline{\partial}$) cases suggest that it may be possible to replace the condition $|u(x)| \leq \varepsilon(\|x\|)$ by a requirment that u should decay rapidly, but not a priori uniformly, on a suitable set of unbounded paths; see, for example, Armitage and Goldstein [2]. Here we confirm that there is indeed a general result of this type.

We now fix some further notation. Let M be an $(n-1)$-dimensional manifold, and let $\Gamma\colon M \times [0, +\infty) \to \Omega$ be a continuous function such that (i) $\Gamma(\omega \times (0, +\infty))$ is open for each open subset ω of M, and (ii) for each $\xi \in M$ the set $\gamma_\xi = \{\Gamma(\xi, t)\colon t \geq 0\}$ is closed and unbounded.

Theorem 1. *There exists a continuous function* $\eta\colon [0, +\infty) \to (0, 1]$ *with the following property. If* u *is a solution of* $P(D)u = 0$ *on* Ω *such that*

$$u(x) = O(\eta(\|x\|)) \qquad (\|x\| \to +\infty,\ x \in \gamma_\xi) \tag{1}$$

for a second category set of ξ *in* M, *then* $u \equiv 0$.

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The category condition is indispensible, at least in the holomorphic case: if S is a first category subset of the unit circle and $\eta: [0, +\infty) \to (0, 1]$ is continuous, then there exists a non-constant entire function f such that $f(rz) = o(\eta(r))$ as $r \to +\infty$ for each $z \in S$ (see Schneider [4, Example 10]).

In proving Theorem 1, we indicate first how η is chosen. Let $\{B_1, B_2, \dots\}$ be a countable base for the topology of M. For the moment let k be a fixed positive integer. By hypothesis, $\Gamma(B_k \times (0, +\infty))$ is open. Moreover, this set has an unbounded connected component, since it contains an unbounded connected set of the form $\Gamma(\{\xi\} \times (0, +\infty))$. Let Ω_k be an unbounded, connected, open subset of $\Gamma(B_k \times (0, +\infty))$ such that $\overline{\Omega}_k \subset \Omega$. By Theorem A, there exists a continuous function $\varepsilon_k: [0, +\infty) \to (0, 1]$ with the property that the zero function is the only solution of $P(D)u = 0$ on Ω_k satisfying $|u(x)| \le \varepsilon_k(\|x\|)$ for all $x \in \Omega_k$. We take $\eta: [0, +\infty) \to (0, 1]$ to be a continuous function such that $\eta \le \varepsilon_k$ on $(k, +\infty)$ for each k.

Now suppose that u is a solution of $P(D)u = 0$ on Ω satisfying (1) for all ξ belonging to a second category subset E of M, and define a function Φ on M by

$$\Phi(\xi) = \sup\{|u(x)|/\eta(\|x\|): x \in \gamma_\xi\}.$$

We claim that Φ is lower semi-continuous on M. To prove this, suppose that $\xi \in M$ and that $A < \Phi(\xi)$. Then there exists $x \in \gamma_\xi$, say $x = \Gamma(\xi, t)$, such that $|u(x)|/\eta(\|x\|) > A$. By the continuity of u, η and Γ, there exists $\delta > 0$ such that $|u(y)|/\eta(\|y\|) > A$ whenever $\|x - y\| < \delta$, and there exists an open neighbourhood N of ξ such that $\|x - \Gamma(\zeta, t)\| < \delta$ for all $\zeta \in N$. Hence

$$\Phi(\zeta) \ge |u(\Gamma(\zeta, t))|/\eta(\|\Gamma(\zeta, t)\|) > A \qquad (\zeta \in N),$$

so Φ is lower semi-continuous at ζ. Now define $A_m = \{\xi \in M: \Phi(\xi) \le m\}$ $(m = 1, 2, \dots)$. By the lower semi-continuity of Φ, each A_m is closed. Clearly $\Phi(\xi) < +\infty$ for each $\xi \in E$, so that $E \subseteq \bigcup_{m=1}^{\infty} A_m$. Since E is second category, some A_k has non-empty interior; this interior contains some B_q. If $x \in \Omega_q$, then $x \in \gamma_\xi$ for some $\xi \in B_q \subseteq A_k$, so that $|u(x)|/\eta(\|x\|) \le \Phi(\xi) \le k$. Hence

$$|u(x)| \le k\eta(\|x\|) \le k\varepsilon_q(\|x\|) \qquad (x \in \Omega_q, \|x\| > q).$$

Since, further, u is bounded on the compact set $\{x \in \overline{\Omega}_q: \|x\| \le q\}$, there exists a positive constant C such that $C|u(x)| \le \varepsilon_q(\|x\|)$ for all $x \in \Omega_q$. From our choice of ε_q, it follows that $u = 0$ on Ω_q. Since u is real-analytic on Ω, we conclude that $u = 0$ on Ω.

References

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Department of Pure Mathematics, Queen's University, Belfast BT7 1NN, Northern Ireland

E-mail address: d.armitage@uk.ac.qub.v2