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ON REPRESENTATIONS OF ELEMENTARY SUBGROUPS OF CHEVALLEY GROUPS OVER ALGEBRAS

YU CHEN

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ABSTRACT. It is shown that every nontrivial linear or projective representation of the elementary subgroup of a Chevalley group over an algebra containing an infinite field must have degree greater than or equal to the square root of the dimension of the corresponding Chevalley-Demazure group scheme adding 1 and the equality emerges only if the Chevalley group is of type A_n for $n \geq 1$.

1. INTRODUCTION AND MAIN THEOREM

Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra, and let Φ be the root system of $\mathfrak{g}$ with respect to a (fixed) Cartan subalgebra $\mathfrak{h}$. We denote by P (resp. P_r) the additive group generated by the weights (resp. roots) of Φ. Then for each sublattice Γ between P and P_r one can construct a Chevalley-Demazure group scheme G associated with Γ and Φ, which is a representable covariant functor from the category of commutative rings with units to the category of groups and whose representing Hopf $\mathbb{Z}$-algebra $\mathbb{Z}[G]$ is an integral domain (cf. [2], [4], and [5]). The dimension of a Chevalley-Demazure scheme G is defined to be the transcendental degree of the fraction field of $\mathbb{Z}[G] \otimes \mathbb{C}$ over the complex number field $\mathbb{C}$. If G is a Chevalley-Demazure group scheme and if R is a commutative ring with a unit, the group $G(R)$ is called a Chevalley group over R. In particular, if R^+ is the additive group of R, then for each root $\alpha \in \Phi$ there is a canonical (exponential) homomorphism (cf. [1, §1.3])

$$u_\alpha \colon R^+ \to G(R).$$

We denote by $U_\alpha(R)$ the subgroup of $G(R)$ consisting of $u_\alpha(r)$ for all $r \in R^+$. The elementary subgroup $E(R)$ of $G(R)$ is by definition the subgroup generated by $U_\alpha(R)$ for all $\alpha \in \Phi$. For example, when $\Gamma = P$ and Φ is of type $A_{n-1} (n \geq 2)$, $G(R)$ is the special linear group $SL_n(R)$ and $E(R)$ is the subgroup generated by all $n \times n$ elementary matrices over R. We call a Chevalley-Demazure group scheme (resp. Chevalley group) simple if its root system is indecomposible.

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Let $G(R)$ be a simple Chevalley group over an algebra R containing an infinite field, and let $E(R)$ be the elementary subgroup of $G(R)$. The purpose of this note is to give a lower bound for the degrees of linear or projective representations of $E(R)$. Our main result is the following:

Theorem. *Let R be an associative and commutative algebra over an infinite field, and let $G(R)$ be a simple Chevalley group over R. If ρ is a nontrivial linear or projective representation of $E(R)$, then*

$$(\deg \rho)^2 \geq \dim G + 1. \tag{1}$$

Moreover, if $(\deg \rho)^2$ is equal to $\dim G + 1$, then G must be of type A_n, where $n = \deg \rho - 1$.

From this theorem the following corollary follows immediately.

Corollary. *Let $G(R)$ be a simple Chevalley group over a commutative integral domain R containing an infinite field. Then the following are equivalent:*

(i) *G is of type A_n for $n \geq 1$.*
(ii) *There exists a nontrivial linear or projective representation ρ of $E(R)$ such that*

$$(\deg \rho)^2 = \dim G + 1.$$

2. Proof of the theorem

To prove our theorem, we need to show some properties of Chevalley groups over a commutative ring R and algebraic groups. Let G be a Chevalley-Demazure group scheme associated with a reduced irreducible root system Φ and a sublattice Γ of the weight lattice P with $\Gamma \supseteq P_r$. We denote by T the maximal torus of G defined by

$$T(R) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, R^*)$$

where R^* is the multiplicative group of the units of R. Let

$$h\colon T(R) \to G(R)$$

be the natural embedding. Then for all $\chi \in T(R)$, $\alpha \in \Phi$, we have (cf. [4])

$$h(\chi)u_\alpha(r)h(\chi)^{-1} = u_\alpha(\chi(\alpha)r) \quad \text{for all } r \in R. \tag{2}$$

Let $\{H_{\alpha_1}, H_{\alpha_2}, \dots, H_{\alpha_n}, X_\alpha | \forall\alpha \in \Phi\}$ be a Chevalley basis of $\mathfrak{g}$, where $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is the set of fundamental roots of Φ and $H_{\alpha_i} \in \mathfrak{h}$ for all $1 \leq i \leq n$. We write H_α for $[X_\alpha, X_{-\alpha}]$ for all $\alpha \in \Phi$, and we suppose that Γ is generated by $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. For each $\alpha \in \Phi$ and $r \in R^*$, let $\chi_{\alpha,r}$ be an element of $T(R)$ defined by

$$\chi_{\alpha,r}(\lambda_i) = r^{\lambda_i(H_\alpha)} \quad \text{for all } 1 \leq i \leq n.$$

Then we have for all $\alpha \in \Phi$ and $r \in R$ (cf. [5])

$$\chi_{\alpha,r}(\alpha) = r^2 \tag{3}$$

and

$$h(\chi_{\alpha,r}) = u_\alpha(r)u_{-\alpha}(-r^{-1})u_\alpha(r)u_{-\alpha}(1)u_\alpha(-1)u_{-a}(1). \tag{4}$$

Lemma 2.1. *Let R be an associative and commutative algebra over a field k containing at least 4 elements. Then $E(R)$ has no proper normal subgroup that contains $E(k)$.*

Proof. Suppose N is a normal subgroup of $E(R)$ such that $N \supseteq E(k)$. Since k^* has at least 3 elements, there exists an element $q \in k^*$ such that $q^2 \neq 1$. Hence by (3) we obtain for all $r \in R$

$$\begin{aligned} u_\alpha(r) &= u_\alpha(q^2(q^2-1)^{-1}r - (q^2-1)^{-1}r) \\ &= u_\alpha(q^2(q^2-1)^{-1}r) \cdot u_\alpha((q^2-1)^{-1}r)^{-1} \\ &= u_\alpha(\chi_{\alpha,q}(\alpha)(q^2-1)^{-1}r) \cdot u_\alpha((q^2-1)^{-1}r)^{-1}. \end{aligned}$$

Note that by the identity (2) we have

$$u_\alpha(\chi_{\alpha,q}(\alpha)(q^2-1)^{-1}r) = h(\chi_{\alpha,q})u_\alpha((q^2-1)^{-1}r)h(\chi_{\alpha,q})^{-1}$$

and by the identity (4) we have

$$h(\chi_{\alpha,q}) \in E(k) \quad \text{for all } \alpha \in \Phi.$$

Hence

$$u_\alpha(r) = h(\chi_{\alpha,q})u_\alpha((q^2-1)^{-1}r)h(\chi_{\alpha,q})^{-1}u_\alpha((q^2-1)^{-1}r)^{-1} \in N.$$

This yields

$$U_\alpha(R) \subseteq N \quad \text{for all } \alpha \in \Phi,$$

which implies that $N = E(R)$.

Lemma 2.2. *Let $G'(K')$ be an absolutely almost simple algebraic group over an algebraically closed field K'. If there exists a homomorphism from $E(k)$ to $G'(K')$ with Zariski dense image, then*

(i) *the root systems of G and G' are isomorphic to each other provided that G' is neither of type B_n nor of type C_n for $n > 2$;*
(ii) $\dim G = \dim G'(K')$.

Proof. See [3, Corollary 2.3 and Corollary 2.4].

Proof of the theorem. Suppose R is a k-algebra, where k is an infinite field. Let $\rho \colon E(R) \to \mathrm{GL}_{n+1}(k')$ (resp. $\mathrm{PGL}_{n+1}(k')$) be a representation over a field k', and let K' be a universal field of k'. It follows from Lemma 2.1 that $\rho(E(k))$ is nontrivial since $\ker\rho$ is a proper normal subgroup of $E(R)$. We show at first that the Zariski closure $\overline{\rho(E(k))}$ of $\rho(E(k))$ in $\mathrm{GL}_{n+1}(K')$ (resp. $\mathrm{PGL}_{n+1}(K')$) is a connected subgroup. Let $\overline{\rho(E(k))}^\circ$ be the connected component of $\overline{\rho(E(k))}$ which contains the identity element of $\mathrm{GL}_{n+1}(K')$ (resp. $\mathrm{PGL}_{n+1}(K')$), and let

$$\delta \colon \overline{\rho(E(k))} \to \overline{\rho(E(k))}/\overline{\rho(E(k))}^\circ$$

be the natural homomorphism. Consider a composition of homomorphisms

$$E(k) \overset{\delta\rho=\beta}{\to} \overline{\rho(E(k))}/\overline{\rho(E(k))}^\circ.$$

Since $\overline{\rho(E(k))}/\overline{\rho(E(k))}^\circ$ is a finite group, we have

$$|E(k)/\ker\beta| < \infty.$$

This means that, since $E(k)$ is infinite and does not contain any proper normal subgroup of finite index (cf. [6]),

$$E(k) \subseteq \ker \beta.$$

Thus we have

$$\rho(E(k)) \subseteq \overline{\rho(E(k))}^{\circ} \subseteq \overline{\rho(E(k))}.$$

By taking Zariski closures of the above groups, we obtain

$$\overline{\rho(E(k))}^{\circ} = \overline{\rho(E(k))}, \tag{5}$$

which means that $\overline{\rho(E(k))}$ is connected.

We show now

$$\dim G \leq \dim \overline{\rho(E(k))}. \tag{6}$$

Let $[E(k), E(k)]$ (resp. $[\rho(E(k)), \rho(E(k))]$) be the commutator subgroup of $E(k)$ (resp. $\rho(E(k))$). Since (cf. [6])

$$E(k) = [E(k), E(k)], \tag{7}$$

we have

$$\overline{\rho(E(k))} = \overline{[\rho(E(k)), \rho(E(k))]} = [\overline{\rho(E(k))}, \overline{\rho(E(k))}],$$

where $\overline{[\rho(E(k)), \rho(E(k))]}$ is the Zariski closure of $[\rho(E(k)), \rho(E(k))]$ while $[\overline{\rho(E(k))}, \overline{\rho(E(k))}]$ is the commutator subgroup of $\overline{\rho(E(k))}$. In particular, $\overline{\rho(E(k))}$ is not a solvable group. Let $\mathfrak{R}$ be the solvable radical of $\overline{\rho(E(k))}$. Then $\overline{\rho(E(k))}/\mathfrak{R}$ is a nontrivial semisimple algebraic group. Suppose $\{G_i\}_{i=1}^{m}$ is the family of the simple components of $\overline{\rho(E(k))}/\mathfrak{R}$, and let G_i^{ad} be an adjoint simple algebraic group of the same type as G_i for $1 \leq i \leq m$. Then there exists an isogeny

$$\varepsilon\colon \overline{\rho(E(k))}/\mathfrak{R} \to \prod_{i=1}^{m} G_i^{\mathrm{ad}}.$$

Let π be the natural morphism from $\overline{\rho(E(k))}$ to $\overline{\rho(E(k))}/\mathfrak{R}$, and let

$$p_j\colon \prod_{i=1}^{m} G_i^{\mathrm{ad}} \to G_j^{\mathrm{ad}}$$

be the canonical projection for $1 \leq j \leq m$. Note that the image of a Zariski dense subset of $\prod_{i=1}^{m} G_i^{\mathrm{ad}}$ (resp. $\overline{\rho(E(k))}/\mathfrak{R}$, $\overline{\rho(E(k))}$) under the map p_j (resp. ε, π) is Zariski dense in G_j^{ad} (resp. $\prod_{i=1}^{m} G_i^{\mathrm{ad}}$, $\overline{\rho(E(k))}/\mathfrak{R}$), hence the image of a Zariski dense subset of $\overline{\rho(E(k))}$ under the composite

$$p_j\varepsilon\pi\colon \overline{\rho(E(k))} \to G_j^{\mathrm{ad}}$$

is also a Zariski dense subset for $1 \leq j \leq m$. In particular, we have for $1 \leq j \leq m$

$$\overline{p_j\varepsilon\pi\rho(E(k))} = p_j\varepsilon\pi(\overline{\rho(E(k))}) = G_j^{\mathrm{ad}},$$

which means that $p_j\varepsilon\pi\rho$ is a homomorphism from $E(k)$ to G_j^{ad} with Zariski dense image. Hence it follows from Lemma 2.2(ii) that for all $1 \leq j \leq m$

$$\dim G = \dim G_j^{\mathrm{ad}}. \tag{8}$$

Since G_j and G_j^{ad} have the same dimension while

$$\dim G_j \leq \dim \overline{\rho(E(k))}/\mathfrak{R} \leq \dim \overline{\rho(E(k))},$$

the identity (8) implies immediately (6).

If ρ is a linear representation, we then have by (7)

$$\rho(E(k)) \subseteq \mathrm{SL}_{n+1}(k'),$$

hence

$$\overline{\rho(E(k))} \subseteq \mathrm{SL}_{n+1}(K').$$

Thus for a linear (resp. projective) representation ρ we have

$$(9) \quad \dim \overline{\rho(E(k))} \leq \dim \mathrm{SL}_{n+1}(K') \quad (\text{resp. } \dim \mathrm{PGL}_{n+1}(K')) = (\deg \rho)^2 - 1$$

from which (1) follows.

Moreover, if $(\deg \rho)^2$ is equal to $\dim G + 1$, it follows from (6) and (9)

$$\dim G = \dim \overline{\rho(E(k))} = \dim \mathrm{SL}_{n+1}(K') \quad (\text{resp. } \dim \mathrm{PGL}_{n+1}(K')),$$

which implies, since $\overline{\rho(E(k))}$ is connected by (5),

$$\overline{\rho(E(k))} = \mathrm{SL}_{n+1}(K') \quad (\text{resp. } \mathrm{PGL}_{n+1}(K')).$$

In other words, ρ induces a homomorphism from $E(k)$ to $\mathrm{SL}_{n+1}(K')$ (resp. $\mathrm{PGL}_{n+1}(K')$) with Zasriski dense image. Hence it follows from Lemma 2.2(i) that G must be of type A_n and $n = \deg \rho - 1$.

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Department of Mathematics, University of Torino, Via Carlo Alberto 10, I-10123 Torino, Italy

E-mail address: yuchen@unimat.polito.it

Institute of System Science, Academia Sinica, Beijing 100080, China