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MEASURES SATISFYING A REFINED DOUBLING CONDITION AND ABSOLUTE CONTINUITY

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ABSTRACT. It is shown that for certain subsets $F \subset \mathbb{R}^n$, two measures with support F satisfying a refined doubling condition are necessarily mutually absolutely continuous. This is contrary to the situation with measures satisfying the usual doubling condition, in which case no such result is available.

1. INTRODUCTION

Let F be a closed nonempty subset of $\mathbb{R}^n$, $0 < s \leq n$, and denote by $B(x, r)$ the open ball with center x and radius r. A positive Borel measure μ with support F satisfies *the condition* D_s if μ is finite on finite sets and there is a constant $c > 0$ such that

$$\mu(B(x, kr)) \leq ck^s\mu(B(x, r)), \qquad x \in F, k \geq 1, kr \leq 1. \tag{1}$$

In this paper we show that if F is a d-set with $d = s$ and μ_1 and μ_2 are measures with support F satisfying D_s, then μ_1 and μ_2 are mutually absolutely continuous. For the definition of a d-set we refer to Section 2, but mention here as examples that the closure of a Lipschitz domain in $\mathbb{R}^n$ is a d-set with $d = n$, and the usual Cantor ternary set is a d-set with $d = \ln 2/\ln 3$. In Section 4 it is pointed out how the result for d-sets can be used to obtain generalizations to more general sets F.

In [4] it is shown that if F is a compact subset of $\mathbb{R}^n$, then there always exists a positive Borel measure μ with support F satisfying the condition D_n and thus in particular the doubling condition

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)), \qquad x \in F, 0 < r \leq 1/2. \tag{2}$$

The question whether two measures with support F which satisfy the doubling condition are mutually absolutely continuous is discussed in [4], and it is noted by referring to an example by A. Beurling and L. Ahlfors that this is not the case even when F is an interval. The results of the present paper show that replacing the doubling condition by the refined doubling condition D_s, we get mutual absolute continuity for classes of closed sets F.

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For the background to the study of measures satisfying the refined doubling condition D_s and for applications of such measures we refer to [2] and [4].

2. Classes of measures

The notation D_s for measures satisfying (1) is the same as in [4], except that in that paper a global condition is used; the condition $kr \leq 1$ is not imposed. It is clear that if μ satisfies the condition D_s, then it satisfies the doubling condition (2). The converse holds in the sense that if μ satisfies the doubling condition, then it satisfies D_s for some s. To see this take an integer i so that $2^{i-1} < k \leq 2^i$, let c be as in (2), and put $s_0 = \ln c/\ln 2$. Then $\mu(B(x, kr)) \leq c^i \mu(B(x, k2^{-i}r)) \leq c^i \mu(B(x, r))$ where $c^i = 2^{is_0} = 2^{s_0}2^{(i-1)s_0} \leq 2^{s_0}k^{s_0}$, so (1) is fulfilled with $s = s_0$. Note also that if μ satisfies (1), then $\mu(B(x, 1)) = \mu(B(x, (1/r)r)) \leq cr^{-s}\mu(B(x, r))$, so the condition D_s implies

$$\mu(B(x, r)) \geq cr^s \mu(B(x, 1)), \qquad 0 < r \leq 1. \tag{3}$$

A measure μ which satisfies

$$c_1 r^d \leq \mu(B(x, r)) \leq c_2 r^d, \qquad x \in F, 0 < r \leq 1, \tag{4}$$

for some positive constants c_1 and c_2 is called a *d-measure* on F. If F is the support of a d-measure, then F is a *d-set.* Any two d-measures on a d-set F are equivalent. If F is a d-set, then the restriction to F of the d-dimensional Hausdorff measure is a d-measure on F. We denote this measure by m and use it as a canonical d-measure on F. For the theory of d-measures, including proofs of the above results, see [3].

We conclude this section by giving an example of a measure satisfying the condition D_n taking $n = 1$ and F as the closed interval $[0, 1]$. Of course, the Lebesgue measure provides such an example, but we want to exhibit a less trivial one. Let the function f be given by $f(x) = 1/\sqrt{x}$, $0 < x < 1$, and $f(x) = 0$ elsewhere, and put $d\mu = f dx$. Then μ satisfies D_1; we sketch a proof of this. Let $x_0 \in [0, 1]$ and $R > 0$. If $R \leq x_0$, then $\mu((x_0 - R, x_0)) \leq \mu(B(x_0, R)) \leq 2\mu((x_0 - R, x_0))$ and $\mu((x_0 - R, x_0)) = 2(\sqrt{x}_0 - \sqrt{x_0 - R}) = 2R/(\sqrt{x}_0 + \sqrt{x_0 - R})$, so

$$R/\sqrt{x}_0 \leq \mu(B(x_0, R)) \leq 4R/\sqrt{x}_0.$$

The latter inequality in this formula holds for $R > x_0$, too, since then $\mu(B(x_0, R)) \leq \mu((0, x_0 + R)) \leq 2\sqrt{x_0 + R} = 2(x_0 + R)/\sqrt{x_0 + R} \leq 4R/\sqrt{x}_0$. Thus, if $r \leq x_0$ and $kr \leq 1$ we have $\mu(B(x_0, kr)) \leq 4kr/\sqrt{x}_0 \leq 4k\mu(B(x_0, r))$. If instead $r > x_0$, then $\mu(B(x_0, kr)) \leq \mu((0, (k+1)r)) \leq 2\sqrt{(k+1)r} = \sqrt{k+1}\mu((0, r)) \leq 2k\mu(B(x_0, r))$, and thus we have shown that μ satisfies D_1.

3. Mutual absolute continuity

It is well known (see, e.g., [4]) that there exists a measure μ defined on $[0, 2\pi]$ which is singular with respect to the one-dimensional Lebesgue measure and satisfies the doubling condition, so two measures with support on a compact set $F \subset \mathbb{R}^n$ satisfying the doubling condition need not be mutually absolutely continuous. Since, as we saw, a measure satisfying the doubling condition also satisfies the condition D_s if s is big enough, this also shows, if we consider the interval $[0, 2\pi]$ as a subset of some $\mathbb{R}^n$ with n big enough, that two measures satisfying the condition D_n on a closed subset $F \subset \mathbb{R}^n$ need not be mutually absolutely continuous. Therefore, in order to get absolute continuity for measures satisfying the refined doubling condition, one has to impose restrictions on F. We work in this section with an n-set F, or more generally, corresponding to the condition D_d, a d-set F, and give some generalizations in the next section.

Assume that F is a d-set with canonical d-measure m, and that μ is a measure with support F satisfying D_d. From (3) it follows that $\mu(B(x, r)) \geq cr^d$, $0 < r \leq 1$, as long as x belongs to some compact set $K \subset F$, for if $\inf_{x\in K} \mu(B(x, 1)) = 0$, then there exists a point $x_0 \in K$ and a sequence $x_k \to x_0$, $k \to \infty$, such that $\mu(B(x_0, 1/2)) \leq \mu(B(x_k, 1)) \to 0$, $k \to \infty$, which contradicts the assumption that μ has support F. Combining this with (4) we get that

$$m(B(x, r)) \leq c\mu(B(x, r)), \qquad x \in K, 0 < r \leq 1.$$

But this means that m is absolutely continuous with respect to μ, $m << \mu$ for short, as can be seen by the following argument. Let $\mu(E) = 0$, take a compact subset K of E, and, given $\epsilon > 0$, an open set $O \supset K$ with $\mu(O) < \epsilon$, and cover K with finitely many balls $B(x_i, r) \subset O$, $i = 1, 2, \dots, i_0$. Take a subset S of these balls such that the balls in S are disjoint and the balls $3B$, $B \in S$, cover K. (This can be achieved by choosing the first ball to be in S as a ball from the covering with largest radius, then omitting all balls from the covering which intersect the first chosen ball, taking as a second ball in S a ball with largest radius among the remaining balls, and so on.) Then we have

$$m(K) \leq \sum_{B\in S} m(3B) \leq c \sum_{B\in S} m(B) \leq c \sum_{B\in S} \mu(B) \leq c\mu(O) \leq c\epsilon,$$

so we must have $m(K) = 0$ and consequently $m(E) = 0$.

Thus we have shown that if F is a d-set with d-measure m and μ is a measure with support F satisfying the condition D_d, then $m << \mu$. The following theorem shows that we also have that $\mu << m$.

Theorem 1. *Let $0 < d \leq n$, and let $F \subset \mathbb{R}^n$ be a d-set with canonical d-measure m. Assume that μ is a measure with support F satisfying the condition D_d. Then μ is absolutely continuous with respect to m.*

Before proving the theorem we note that in view of the discussion above the following corollary holds.

Corollary 1. *Let $0 < d \leq n$, and let $F \subset \mathbb{R}^n$ be a d-set. Assume that μ_1 and μ_2 are measures with support F satisfying the condition D_d. Then μ_1 and μ_2 are mutually absolutely continuous.*

Proof of Theorem 1. Put

$$D\mu(x) = \lim_{r\to 0} \frac{\mu(B(x, r))}{m(B(x, r))}$$

at every point in F where the limit exists, and call the limit $f(x)$ at those points. Then $D\mu$ exists m-a.e. and the function f is locally integrable with respect to m; see, e.g., [1] (actually f equals the Radon-Nikodym derivative of the part of μ which is absolutely continuous with respect to m). Put $A = \{x \in F : D\mu(x) \text{ exists}\}$.

Let $x_0 \in A$. Then, given $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{\mu(B(x_0, r))}{m(B(x_0, r))} - f(x_0)\right| < \epsilon, \qquad r \leq \delta,$$

and hence

$$\mu(B(x_0, r)) < m(B(x_0, r))(f(x_0) + \epsilon), \qquad r \leq \delta. \tag{5}$$

For $\delta < r \leq 1$ we get, using that μ satisfies D_d, the formula (5), and that m is a d-measure, $\mu(B(x_0, r)) = \mu(B(x_0, (r/\delta)\delta)) \leq c(r/\delta)^d \mu(B(x_0, \delta)) \leq c(r/\delta)^d m(B(x_0, \delta))(f(x_0) + \epsilon) \leq c(r/\delta)^d \delta^d (f(x_0) + \epsilon)$, and so since m is a d-measure,

$$\mu(B(x_0, r)) \leq cm(B(x_0, r))(f(x_0) + \epsilon), \qquad \delta < r \leq 1.$$

Combining this inequality with (5) we see that it holds for $0 < r \leq 1$, and letting $\epsilon \to 0$ we get

$$\mu(B(x_0, r)) \leq cm(B(x_0, r))f(x_0), \qquad 0 < r \leq 1, \, x_0 \in A. \tag{6}$$

Now let $E \subset F$ be a set with $m(E) = 0$. We shall show that $\mu(E) = 0$, which proves the theorem.

We assume first that F is compact. Put $A_N = \{x \in A \cap F : f(x) \leq N\}$, and fix an $\epsilon > 0$. Choose $N = N(f, \epsilon) > 0$ so big that A_N is nonempty and $\int_{F \setminus A_N} f dm < \epsilon$; then $Nm(F \setminus A_N) \leq \int_{F \setminus A_N} f dm < \epsilon$, so

$$m(F \setminus A_N) < \epsilon/N.$$

Next take K compact satisfying $K \subset E$ and $\mu(K) \geq \frac{1}{2}\mu(E)$, and choose an open set O so that $K \subset O$ and

$$m(O) < \epsilon/N.$$

Cover K with balls contained in O with centers in K, and choose a finite subcovering consisting of distinct balls $B_\nu = B(x_\nu, r_\nu)$, $\nu = 1, 2, \ldots, \nu_0$. Let

$$r'_\nu = \max(r_\nu, \sup\{r : B(x_\nu, r) \cap A_N = \varnothing\})$$

and put $C_\nu = B(x_\nu, r'_\nu)$ so that $B_\nu \subset C_\nu$. Choose $x'_\nu \in A_N$ in $2C_\nu$ and let $C'_\nu = B(x'_\nu, 5r'_\nu)$; then $3C_\nu \subset C'_\nu$.

Choose now a ball with largest radius among the balls C_ν, $\nu = 1, 2, \ldots, \nu_0$, call it S_1, and denote the corresponding C'_ν by S'_1. Put $\mathscr{M}_1 = \{C_\nu : C_\nu \cap S_1 \neq \varnothing\}$. Then our choice of S_1 guarantees that $C_\nu \subset S'_1$ if $C_\nu \in \mathscr{M}_1$. Choose next S_2 as a ball with largest radius among the balls $C_\nu \notin \mathscr{M}_1$, and put $\mathscr{M}_2 = \{C_\nu \notin \mathscr{M}_1 : C_\nu \cap S_2 \neq \varnothing\}$, and so on. Thus, in the ith step, $i > 1$, we choose S_i as the ball with largest radius among the balls $C_\nu \notin \cup_{j=1}^{i-1} \mathscr{M}_j$ and put $\mathscr{M}_i = \{C_\nu \notin \cup_{j=1}^{i-1} \mathscr{M}_j : C_\nu \cap S_i \neq \varnothing\}$. The procedure stops at some $i \leq \nu_0$, say at i_0. Then the balls S_i are disjoint and the balls S'_i, $i = 1, 2, \ldots, i_0$, cover K. Let $I_1 = \{i, 1 \leq i \leq i_0 : S_i \text{ equals some } B_\nu\}$ and $I_2 = \{i, 1 \leq i \leq i_0 : i \notin I_1\}$. Then

$$\sum_{i \in I_1} m(S_i) < m(O) < \epsilon/N$$

since $S_i \subset O$ if $i \in I_1$ and

$$\sum_{i\in I_2} m(S_i) < m(F \setminus A_N) < \epsilon/N$$

since by construction the balls S_i, $i \in I_2$, are all in the complement of A_N.

Now, since $x'_\nu \in A_N$, and because we may assume that things have been arranged in such a way that the radii r'_ν are all small, we get from (6) that

$$\mu(S'_i) \le cNm(S'_i).$$

Consequently, since $S'_i \subset 7S_i$ and m satisfies the doubling condition,

$$\sum_i \mu(S'_i) \le cN \sum_i m(S'_i) \le cN \sum_i m(S_i) \le cN(2\epsilon/N) = c\epsilon,$$

so $\mu(K) \le c\epsilon$, where c does not depend on ϵ. Thus we must have $\mu E = 0$ since $\mu(K) \ge \frac{1}{2}\mu(E)$.

If F is not compact, one has to do some modifications. Let $X = F \cap \{x : |x| \le R\}$ and $X_1 = F \cap \{x : |x| \le R+1\}$, where $R > 0$ is chosen so big that X has positive m-measure. It is enough to prove that $\mu(X \cap E) = 0$. This time let $A_N = \{x \in A \cap X_1 : f(x) \le N\}$ and choose N so that $m(X_1 \setminus A_N) < \epsilon/N$. Take a compact set $K \subset X \cap E$ and proceed as before. Then the balls S_i, $i \in I_2$, are all in $\{x : |x| \le R+1\} \setminus A_N$ and the proof works as before.

4. An example

Theorem 1 can be generalized to more general sets F than d-sets. Let F be a closed subset of $\mathbb{R}^n$, let μ be a measure with support F satisfying the condition D_d, and let m be the restriction to F of the d-dimensional Hausdorff measure. Suppose that closed subsets F_k, $k = 1, 2, \ldots$, of F can be chosen that satisfy the following conditions:

(1) $F_k \subset F_{k+1}$, $k = 1, 2, \ldots$,
(2) the sets F_k are d-sets,
(3) the measures $\mu_k = \mu|F_k$ satisfy the condition D_d, and
(4) the set $G := F \setminus F_0$ where $F_0 = \cup_{k=1}^{\infty} F_k$ has measure zero with respect to the measures μ and m.

Let E be a set with $m(E) = 0$. Then $\mu(E) = \mu(E \cap G) + \mu(E \cap F_0) = \mu(E \cap F_0) = \lim_{k\to\infty} \mu(E \cap F_k) = 0$, where the last equality follows from Theorem 1 since $m(E \cap F_k) = 0$. Thus $\mu << m$, and in the same way one shows $m << \mu$. We exemplify how to use these observations in an explicit example.

Example. Let $\gamma > 1$, and let $F \subset \mathbb{R}^2$ be given by $F = \{(x_1, x_2) : 0 \le x_1 \le 1, 0 \le x_2 \le x_1^\gamma\}$. Then F is not a d-set for some d. Take F_k as $\{x \in F : x_1 \ge 1/(k+2)\}$, let m be the Lebesgue measure, and let μ be a measure with support F satisfying D_2. We shall see that the conditions (1)–(4) above are satisfied with $d = 2$. Thus $\mu << m$, and if μ_1 and μ_2 are two measures with support F, then they are mutually absolutely continuous.

Clearly (1) holds and (2) follows since F_k is the closure of a Lipschitz domain. To check (3), take $x \in F_k$. Let $\mu_k = \mu|F_k$ and $kr \le 1$. Then $\mu_k(B(x, kr)) \le \mu(B(x, kr)) \le ck^2\mu(B(x, r/4))$. If $B(x, r/4)$ does not intersect $F \setminus F_k$ we are done, since then

$$\mu(B(x, r/4)) = \mu_k(B(x, r/4)) \le \mu_k(B(x, r)).$$

If $B(x, r/4)$ intersects $F \setminus F_k$ we use that the ball $B(x_0, r/8)$, $x_0 = x + (r/8, 0)$, does not intersect $F \setminus F_k$. Then $\mu(B(x, r/4)) \leq \mu(B(x_0, r/2)) \leq c\mu(B(x_0, r/8)) = c\mu_k(B(x_0, r/8)) \leq c\mu_k(B(x, r))$, which is what we want.

Finally (4) is seen in the following way. Since $\mu(\{x : 0 < x_1 < 1/k\}) \to 0$, $k \to \infty$, and $\mu\{(0, 0)\} \leq \mu(B(p_k, 1/k)) \leq c\mu(B(p_k, 1/(2k))) \leq c\mu(\{x : 0 < x_1 < 1/k\})$ where $p_k = (1/(2k), 0)$, we must have $\mu\{(0, 0)\} = 0$.

References

1. A.S. Besicovitch, *A general form of the covering principle and relative differentiation of additive functions*, Proc. Cambridge Philos. Soc. **41** (1945), 103–110; II, Proc. Cambridge Philos. Soc. **42** (1946), 1–10.
2. A. Jonsson, *Besov spaces on closed subsets of* $\mathbb{R}^n$, Trans. Amer. Math. Soc. **341** (1994), 355–370.
3. A. Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Mathematical Reports, vol. 2, Part 1, Harwood Acad. Publ., Chur, Switzerland, 1984.
4. A.L. Volberg and S.V. Konyagin, *On measures with the doubling condition*, Izv. Akad. Nauk CCCP Ser. Mat. **51** (1987); English transl., Math USSR-Izv. **30** (1988), 629–637.

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