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A NEW SIMPLE PROOF OF THE GELFAND-MAZUR-KAPLANSKY THEOREM

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ABSTRACT. We provide an almost purely algebraic proof of Kaplansky's refinement of the Gelfand-Mazur theorem asserting that the reals, complex, and quaternions are the only associative normed real algebras with no nonzero topological divisors of zero.

1. INTRODUCTION

The Gelfand-Mazur theorem asserts that, if A is an associative normed real division algebra, then A is isomorphic to the reals, complex, or quaternions [1, Theorem 14.7]. Of course, the assumption that the associative normed algebra A is a division algebra is formally stronger than that A has no nonzero topological divisors of zero. However, I. Kaplansky proved in [5] that this last condition actually implies the former. In this note we will provide a very simple proof of Kaplansky's theorem, and we will slightly improve its statement by weakening the assumption of absence of topological divisors of zero to the absence of "joint" topological divisors of zero.

A summary of our proof is the following. With the help of Frobenius's theorem on quadratic division associative real algebras, we will reduce the problem to the case in which A is commutative. Then, as any associative commutative algebra with no nonzero divisors of zero, A imbeds naturally in its extended centroid, which is a field extension of $\mathbb{R}$. Because of the absence of topological divisors of zero, we will see that the extended centroid of A can be provided with an algebra norm, and then the Gelfand-Mazur theorem will conclude the proof.

Most of this argument can be developed without any additional effort in a general nonassociative context, and we will do so in §2. Section 3 will then contain the conclusion of the proof of the main result.

2. TOPOLOGICAL DIVISORS OF ZERO AND EXTENDED CENTROID

A nonassociative algebra A is said to be prime if the product of two arbitrary nonzero ideals of A is nonzero. A partially defined centralizer on the prime algebra A will be a linear mapping (say f) from a nonzero ideal of A (say

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$\mathrm{dom}(f)$) into A satisfying

$$f(xy) = f(x)y \quad \text{and} \quad f(yx) = yf(x)$$

for all x in $\mathrm{dom}(f)$ and y in A. The relation $\simeq$ defined on the set of all partially defined centralizers on A by $f \simeq g$ if and only if f and g coincide on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ is an equivalence relation. The sum and composition of two partially defined centralizers on A, as partially defined operators, are also partially defined centralizers on A. These operations are compatible with the above equivalence relation, and the extended centroid $C(A)$ of A is defined as the quotient set with the induced operations. By [4, Theorem 2.1], the extended centroid $C(A)$ of the prime algebra A is a field extension of the base field for A and, in particular, is commutative.

Now we will follow, with minor variants, some ideas in [7] and [2].

Lemma 1. *Let A be a normed prime real algebra, and assume that every partially defined centralizer on A is continuous. Then the extended centroid of A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Proof. In view of the Gelfand-Mazur theorem, it is enough to provide $C(A)$ with an algebra norm. In fact we will see that, if for α in $C(A)$ we define

$$\|\alpha\| := \inf\{\|f\| : f \in \alpha\},$$

then $\|\cdot\|$ becomes an algebra norm on $C(A)$. The property $\|\lambda\alpha\| = |\lambda|\|\alpha\|$, for λ in $\mathbb{R}$ and α in $C(A)$, is clear. Let α, β be in $C(A)$. For arbitrary $f \in \alpha$ and $g \in \beta$, consider the mapping h (respectively, k) from $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ (respectively, $\{x \in \mathrm{dom}(g) : g(x) \in \mathrm{dom}(f)\}$) into A given by $h(x) := f(x) + g(x)$ (respectively, $k(x) := f(g(x))$). Then h and k are partially defined centralizers on A with $h \in \alpha + \beta$, $k \in \alpha\beta$, $\|h\| \leq \|f\| + \|g\|$, and $\|k\| \leq \|f\|\|g\|$. Hence

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\| \quad \text{and} \quad \|\alpha\beta\| \leq \|\alpha\|\|\beta\|.$$

Now $\|\cdot\|$ is an algebra seminorm on $C(A)$ satisfying $\|\mathbf{1}\| = 1$ (where $\mathbf{1}$ denotes the unit of $C(A)$), hence an algebra norm because $C(A)$ is a field. □

Following [1, Definition 2.12], given a normed algebra A, we will say that an element x in A is a joint topological divisor of zero in A if there is a nonnull sequence $\{y_n\}$ in A satisfying $\{xy_n\} \to 0$ and $\{y_n x\} \to 0$. Equivalently, an element x in A is not a joint topological divisor of zero in A if there exists a positive number m satisfying

$$m\|y\| \leq \|xy\| + \|yx\|$$

for all y in A.

Theorem 1. *Let A be a normed real algebra, and assume that every nonzero ideal of A contains an element which is not a joint topological divisor of zero in A. Then A is prime and the extended centroid of A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Proof. Elements x in A satisfying $x^2 = 0$ are joint topological divisors of zero; hence from the assumption on A it follows that, if P is an ideal of A and if $P^2 = 0$, then $P = 0$. For P and Q ideals of A, from the inclusions $QP \subseteq P \cap Q$ and $(P \cap Q)^2 \subseteq PQ$ and the above observation it follows that $QP = 0$ whenever $PQ = 0$. Therefore, if $PQ = 0$ and if $P \neq 0$, then every element of Q is a joint topological divisor of zero in A, and so $Q = 0$. Now that we have shown that A is prime, the proof will be concluded by invoking Lemma 1 and proving that every partially defined centralizer on A is continuous. But, if f is such a partially defined centralizer, then there exist x in $\operatorname{dom}(f)$ and a positive number m such that $m\|y\| \leq \|xy\| + \|yx\|$ for all y in A. Now, for every y in $\operatorname{dom}(f)$, we have

$$m\|f(y)\| \leq \|xf(y)\| + \|f(y)x\| = \|f(x)y\| + \|yf(x)\| \leq 2\|f(x)\|\|y\|.$$

Hence f is continuous. □

The following corollary is immediate.

Corollary 1. *Let A be a normed real algebra with no nonzero joint topological divisors of zero. Then A is prime and the extended centroid of A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Remark 1. A prime algebra is called centrally closed if its extended centroid coincides with the base field. It follows from the results above that normed complex algebras whose nonzero ideals contain elements which are not joint topological divisors of zero in the whole algebra are centrally closed prime algebras. As a consequence, normed complex algebras without nonzero joint topological divisors of zero are prime and centrally closed.

3. Revisiting the Gelfand-Mazur-Kaplansky theorem

Now, with the help of the ideas developed above, the Gelfand-Mazur-Kaplansky theorem follows easily.

Theorem 2. *If A is an associative normed real algebra with no nonzero joint topological divisors of zero, then A is isomorphic to the reals, complex, or quaternions.*

Proof. First assume additionally that A is commutative. Then, identifying each element of A with the operator on A consisting in multiplying by the given element, elements of A can be regarded as (everywhere defined) centralizers on A, and passing to the quotient by the equivalence of partially defined centralizers, we have a natural embedding $A \hookrightarrow C(A)$. By Corollary 1, A is isomorphic to a real subalgebra of $\mathbb{R}$ or $\mathbb{C}$; hence A itself is isomorphic to $\mathbb{R}$ to $\mathbb{C}$. For the general case, first note that, from the above, A has nonzero idempotents, namely, the unit elements of its one-generated subalgebras. Let e be a nonzero idempotent in A; then, again by the absence of joint topological divisors of zero in A, we have $eA(1-e) = (1-e)Ae = (1-e)A(1-e) = 0$ (observe that $x^2 = 0$ whenever x is in $[eA(1-e)] \cup [(1-e)Ae]$, while $xe = ex = 0$ whenever x is in $(1-e)A(1-e))$. It follows that $A = eAe$, and therefore e is the (unique) unit element of A. Now A is a quadratic division associative real algebra so, by Frobenius's theorem (see, for example, [3, p. 229]), it is isomorphic to the reals, complex, or quaternions. □

Replacing in the above argument Frobenius's classical associative theorem by its alternative extension (see, for example, [3, p. 262]), we obtain in an analogous way the following slight improvement of [6, Theorem 3.2].

Theorem 3. *If A is an alternative normed real algebra with no nonzero joint topological divisors of zero, then A is isomorphic to the reals, complex, quaternions, or octonions.*

Remark 2. Because the proof of Theorem 2 needed Corollary 1 only in the commutative case (in which joint topological divisors of zero and topological divisors of zero are the same), weaker (hence slightly easier) versions of that corollary and of the previous Theorem 1 would have been enough.

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