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MEASURABILITY OF UNIONS OF CERTAIN DENSE SETS

ALBERTO ALONSO AND JAVIER F. ROSENBLUETH

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ABSTRACT. In this paper we study measurability properties of sets of the form

$$E_t = \{t + m\alpha_1 + n\alpha_2 | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R})$$

where α_1, α_2 are given real numbers with α_1/α_2 irrational. Sets such as these have played an important role to establish certain fundamental results in measure theory. However, the question of measurability of unions of these sets seems not to have been solved. In an initial guess, no sets C and T seem apparent for which $0 < mA < mT$, where m denotes the Lebesgue measure in $\mathbf{R}$ and $A = \bigcup_{t\in C} E_t \cap T$. In fact, we prove that if T is any Lebesgue measurable subset of $\mathbf{R}$, then no such sets can exist: no matter which C we choose, if A is measurable then mA equals 0 or mT. Moreover, if A is a nonmeasurable set, the same applies to its Lebesgue outer measure. However, if we remove the condition on T of being measurable, we provide an example of (nonmeasurable) sets C and T for which the outer measure of A lies in between 0 and the outer measure of T.

1. INTRODUCTION

Let α_1, α_2 be given real numbers with α_1/α_2 irrational, and consider the sets

$$E_t := \{t + m\alpha_1 + n\alpha_2 | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R}).$$

The problem we shall be concerned with in this paper is the existence of sets $C, T \subset \mathbf{R}$ for which $0 < mA < mT$, where $A = \bigcup_{t\in C} E_t \cap T$ and m denotes the Lebesgue measure in $\mathbf{R}$.

Sets such as these have played an important role to establish certain fundamental results in measure theory. For example, it is well known that, if we make use of the axiom of choice and let P be a set which contains exactly one element from each equivalence class under the relation $x \sim y \Leftrightarrow x - y \in E_0$, then the set

$$\bigcup_{t\in P} \{t + m\alpha_1 + 2n\alpha_2 | m, n \in \mathbf{Z}\}$$

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is nonmeasurable and, moreover, each Lebesgue measurable set that is included in it or in its complement has Lebesgue measure zero (see, for example, [1, 2]).

However, the question of measurability of sets of the form $\bigcup_{t\in C} E_t \cap T$ that we have posed above seems not to have been solved.

Observe first that the sets E_t are countable, and thus measurable with measure zero. Also, they are dense in $\mathbf{R}$. This fact can be easily seen as follows. Assume, without loss of generality, that $t = 0$ and $\alpha_1 > 0$. Note that, for every $i \in \mathbf{N}$, there exists a unique $m_i \in \mathbf{Z}$ such that

$$x_i := m_i\alpha_1 + i\alpha_2 \in (0, \alpha_1).$$

Let I be any open interval, set $\varepsilon := mI > 0$, and let $k \in \mathbf{N}$ be such that $k\varepsilon > \alpha_1$. Then, among the $k+1$ numbers $x_1, \dots, x_{k+1}$ in the interval $(0, \alpha_1)$, there must be at least two, say x_i and x_j, such that $|x_i - x_j| < \varepsilon$. Thus, some integral multiple of $x_i - x_j$, i.e., some element of E_0, belongs to the interval I. This shows that E_0 is dense in $\mathbf{R}$, as claimed.

Now, observe that if C is any countable subset of $\mathbf{R}$, then $mA = 0$. On the other hand, if C is any interval and T any measurable set, then $mA = mT$. In an initial guess, no sets C and T seem apparent for which $0 < mA < mT$.

The main result we prove in this paper is the following: if C and T are any subsets of $\mathbf{R}$ with T measurable, then, necessarily, m^*A equals 0 or mT, where m^* denotes the Lebesgue outer measure in $\mathbf{R}$, but if we remove the condition on T of being measurable, the existence of sets C and T such that $0 < m^*A < m^*T$ can be assured.

2. The case T an interval

The purpose of this section is to prove that if C is any subset of $\mathbf{R}$ and T is any interval, then, necessarily, m^*A equals 0 or mT, where $A = \bigcup_{t\in C} E_t \cap T$. Without loss of generality, we shall prove this result assuming that $T = [0, 1)$.

For all x and y real numbers, define the *sum modulo* 1 of x and y by

$$x \oplus y := x + y - n \quad \text{if } x + y \in [n, n+1),\ n \in \mathbf{Z},$$

and consider the sets

$$A_t := \{t \oplus (m\alpha_1 + n\alpha_2) | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R})$$

where α_1 and α_2 are given real numbers with α_1/α_2 irrational.

It is clear that, for any $t \in \mathbf{R}$, $E_t \cap [0, 1) \subset A_t$, but a simple example shows that the other inclusion may not hold. If $\alpha_1 = \sqrt{2}$, $\alpha_2 = 2/9$, and $x = \alpha_1 \oplus \alpha_2$, then $x \in A_0$ and $x = \sqrt{2} - 7/9$. Suppose that $x \in E_0 \cap [0, 1)$ and let $m, n \in \mathbf{Z}$ be such that $x = m\alpha_1 + n\alpha_2$. Then

$$\sqrt{2} - 7/9 = m\sqrt{2} + 2n/9.$$

If $m = 1$ then $n = -7/2$, and if $m \neq 1$ then $\sqrt{2} = (2n+7)/9(1-m)$, relations which are not possible. We conclude that $A_0 \neq E_0 \cap [0, 1)$.

One readily verifies that these sets are related according to

$$A_t = \bigcup_{r\in\mathbf{Z}} E_{t+r} \cap [0, 1).$$

In particular, if α_1 or α_2 is 1, then $A_t = E_t \cap [0, 1)$. Observe that we can consider only this case, without any loss of generality, since if we multiply the sets E_t by $1/\alpha_1$ or $1/\alpha_2$, we will deal with sets of the form

$$\{t + m + n\alpha | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R})$$

with α an irrational number.

Throughout this section we shall use the following notation. If B is a subset of $[0, 1)$, we define the translate modulo 1 of B to be the set

$$B \oplus s := \{x \in [0, 1) | x = b \oplus s \text{ for some } b \in B\} \qquad (s \in \mathbf{R}).$$

For all $s \in \mathbf{R}$, $\mathscr{T}_s$ will denote the operator defined by

$$(\mathscr{T}_s f)(x) = f(x \ominus s)$$

for all functions $f\colon [0, 1) \to \mathbf{C}$ where, for all $x, y \in \mathbf{R}$, $x \ominus y := x \oplus (-y)$. We shall use the same notation for the induced operator on $L^2[0, 1)$. Moreover, we define $\mathscr{T}_s$ acting on subsets of $[0, 1)$ by $\mathscr{T}_s B = B \oplus s$. We shall later justify this last notation by showing that $\mathscr{T}_s \chi_B = \chi_{B \oplus s}$ where χ_B denotes the characteristic function of the set B. Finally, the $L^2[0, 1)$-norm will be denoted by $\| \cdot \|$ and the inner product on $L^2[0, 1)$ by $\langle \cdot, \cdot \rangle$.

Let us begin by establishing some properties of these operators.

2.1 **Lemma.** *For all* $f\colon [0, 1) \to \mathbf{C}$ *and* $t, r \in \mathbf{R}$, *the following hold*:
(a) $\mathscr{T}_t \mathscr{T}_r f = \mathscr{T}_{t+r} f$.
(b) $\mathscr{T}_0 = I$.
(c) $\mathscr{T}_t^{-1} f = \mathscr{T}_{-t} f$.
Moreover, the family $\{\mathscr{T}_s\}_{s \in [0, 1)}$ *of operators in* $L^2[0, 1)$ *is a strongly continuous uniparametric family of unitary operators.*

Proof. (a): Given $x, r, t \in \mathbf{R}$, let $m, n, k \in \mathbf{Z}$ be such that

$$(x \ominus t) - r \in [m, m+1), \quad x - (t + r) \in [n, n+1), \quad x - t \in [k, k+1).$$

Note that $m \le x - (t + r) - k < m + 1$, and so $n = k + m$. Therefore,

$$(x \ominus t) \ominus r = x - (t + r) - (k + m) = x \ominus (t + r).$$

This identity implies that, if $f\colon [0, 1) \to \mathbf{C}$, then

$$(\mathscr{T}_t(\mathscr{T}_r f))(x) = (\mathscr{T}_r f)(x \ominus t) = f((x \ominus t) \ominus r) = f(x \ominus (t + r)) = (\mathscr{T}_{t+r} f)(x).$$

(b): It is clear.

(c): It follows by observing that $\mathscr{T}_t \mathscr{T}_{-t} = \mathscr{T}_{-t} \mathscr{T}_t = \mathscr{T}_0 = I$.

To prove the last assertion, let $f \in L^2[0, 1)$. Then, for all $s \in [0, 1)$,

$$\begin{aligned}
\int_{[0,1)} |(\mathscr{T}_s f)(x)|^2 dx &= \int_{[0,1)} |f(x \ominus s)|^2 \, dx \\
&= \int_{[0,s)} |f(x \ominus s)|^2 \, dx + \int_{[s,1)} |f(x \ominus s)|^2 \, dx \\
&= \int_{[0,s)} |f(x - s + 1)|^2 \, dx + \int_{[s,1)} |f(x - s)|^2 \, dx \\
&= \int_{[1-s,1)} |f(x)|^2 \, dx + \int_{[0,1-s)} |f(x)|^2 \, dx \\
&= \int_{[0,1)} |f(x)|^2 \, dx.
\end{aligned}$$

Thus, $\mathscr{F}_s\colon L^2[0,1)\to L^2[0,1)$, and preserves the norm. Clearly, $\mathscr{F}_s$ is linear and thus it is unitary. To prove continuity, let $[a,b)\subset[0,1)$. From the definition it is clear that

$$\lim_{s\to 0}(\mathscr{F}_s\chi_{[a,b)})(x)=\chi_{[a,b)}(x)\quad \text{if } x\neq a \text{ and } x\neq b$$

so that the convergence is almost everywhere. Also, we have

$$\|\mathscr{F}_s\chi_{[a,b)}\|=\|\chi_{[a,b)\oplus s}\|\le 1.$$

Now, let $g\in L^2[0,1)$. By the dominated convergence theorem, it follows that

$$\lim_{s\to 0}\int \mathscr{F}_s\chi_{[a,b)}(x)g(x)\,dx=\int \chi_{[a,b)}(x)g(x)\,dx.$$

Hence,

$$\lim_{s\to 0}\langle\chi_{[a,b)},\mathscr{F}_s g\rangle=\lim_{s\to 0}\langle\mathscr{F}_s^{-1}\chi_{[a,b)},g\rangle=\lim_{s\to 0}\langle\mathscr{F}_{-s}\chi_{[a,b)},g\rangle=\langle\chi_{[a,b)},g\rangle.$$

We deduce that, if f is any step function, then $\langle f,F_s g\rangle\to\langle f,g\rangle$, $s\to 0$. Let $\varepsilon>0$ and let f be a step function such that $\|f-g\|<\varepsilon/2\|g\|$. Since

$$\begin{aligned}|\langle g,\mathscr{F}_s g\rangle-\langle g,g\rangle|\le{}&|\langle g,\mathscr{F}_s g\rangle-\langle f,\mathscr{F}_s g\rangle|\\&+|\langle f,\mathscr{F}_s g\rangle-\langle f,g\rangle|+|\langle f,g\rangle-\langle g,g\rangle|,\end{aligned}$$

we have

$$\begin{aligned}\overline{\lim_{s\to 0}}\,|\langle g,\mathscr{F}_s g\rangle-\langle g,g\rangle|&\le\overline{\lim_{s\to 0}}(\|g-f\|\cdot\|\mathscr{F}_s g\|)+\|f-g\|\cdot\|g\|\\&=2\|g\|\cdot\|f-g\|<\varepsilon\end{aligned}$$

and so $\langle g,\mathscr{F}_s g\rangle\to\langle g,g\rangle$, $s\to 0$. The result now follows observing that

$$\begin{aligned}\lim_{s\to 0}\|\mathscr{F}_s g-g\|^2&=\lim_{s\to 0}(\|\mathscr{F}_s g\|^2+\|g\|^2-\langle\mathscr{F}_s g,g\rangle-\langle g,\mathscr{F}_s g\rangle)\\&=\|g\|^2+\|g\|^2-2\|g\|^2=0\end{aligned}$$

and this completes the proof of the lemma.

Now, observe that, for any $x,s\in\mathbf{R}$ and $B\subset[0,1)$, the following relations hold:

$$\begin{aligned}x\ominus s\in B&\Leftrightarrow x-s+r\in B\quad\text{for some } r\in\mathbf{Z}\\&\Leftrightarrow x=b+s-r\quad\text{for some } b\in B,\ r\in\mathbf{Z}\\&\Leftrightarrow x\in B\oplus s.\end{aligned}$$

This implies that

$$(\mathscr{F}_s\chi_B)(x)=\chi_B(x\ominus s)=\chi_{B\oplus s}(x)$$

and so the notation $\mathscr{F}_s B=B\oplus s$ is clearly justified.

The following lemma shows that any measurable subset of $[0,1)$ which is invariant under translation modulo 1 for all points in a dense subset of $[0,1)$, must have measure 0 or 1.

2.2 **Lemma.** *Let D be any dense subset of* $[0, 1)$ *and B any measurable subset of* $[0, 1)$ *such that $B \oplus s = B$ for all $s \in D$. Then $mB = 0$ or $mB = 1$.*

Proof. Suppose that $mB \neq 0$ and let $\widetilde{B} := [0, 1) \sim B = \{x \in [0, 1) | x \notin B\}$. The result will follow if we show that $m\widetilde{B} = 0$. For this purpose, let $\varepsilon > 0$ and let $U \supset \widetilde{B}$ be an open set such that $mU < m\widetilde{B} + \varepsilon mB$. Let us assume, for the moment, that $m(U \cap (0, 1) \cap B) = m(U \cap (0, 1)) \cdot mB$. Then

$$\begin{aligned} m\widetilde{B} + \varepsilon mB > mU &\geq \int \chi_U(x)\chi_{[0,1)}(x)\,dx \\ &= \int \chi_{U\cap(0,1)}(x)\chi_B(x)\,dx + \int \chi_U(x)\chi_{\widetilde{B}}(x)\,dx \\ &= m(U \cap (0, 1) \cap B) + m(U \cap \widetilde{B}) \\ &= m(U \cap (0, 1)) \cdot mB + m\widetilde{B}. \end{aligned}$$

Thus, $m\widetilde{B} + \varepsilon mB > (mB + 1)m\widetilde{B}$. But then $\varepsilon > m\widetilde{B}$, and so $m\widetilde{B} = 0$.

The lemma will then be established if we prove the assumption we made: we are going to show that $m(O \cap B) = m(O)m(B)$ if O is any open subset of $[0, 1)$.

To begin with, let us prove that $\mathscr{T}_s B = B$ for a.a. s in $[0, 1)$. Indeed, if $s \in [0, 1)$ and $\{s_n\} \subset D$ converges to s, then

$$\begin{aligned} \|\mathscr{T}_s\chi_B - \chi_B\| &\leq \|\mathscr{T}_s\chi_B - \mathscr{T}_{s_n}\chi_B\| + \|\mathscr{T}_{s_n}\chi_B - \chi_B\| \\ &= \|\mathscr{T}_s\chi_B - \mathscr{T}_{s_n}\chi_B\| \end{aligned}$$

and, in view of Lemma 2.1, $\|\mathscr{T}_s\chi_B - \mathscr{T}_{s_n}\chi_B\| \to 0$, $n \to \infty$. Thus, $\mathscr{T}_s\chi_B = \chi_B$ a.e. Now, with the help of this result, let us prove that $m([a, b) \cap B) = m([a, b))m(B)$ where $[a, b) \subset [0, 1)$. Suppose first that $[a, b)$ has length $1/n$ with $n \in \mathbf{N}$. Then, $\mathscr{T}_{-a}[a, b) = [0, 1/n)$. Also,

$$\begin{aligned} m([a, b) \cap B) &= \int \chi_{[a,b)}(x)\chi_B(x)\,dx = \langle \chi_{[a,b)}, \chi_B \rangle \\ &= \langle \mathscr{T}_{-a}\chi_{[a,b)}, \mathscr{T}_{-a}\chi_B \rangle = \langle \chi_{[0,1/n)}, \chi_B \rangle \\ &= m([0, 1/n) \cap B). \end{aligned}$$

Since

$$\begin{aligned} mB &= \sum_{k=0}^{n-1} m([k/n, (k+1)/n) \cap B) \\ &= \sum_{k=0}^{n-1} m([0, 1/n) \cap B) = n \cdot m([0, 1/n) \cap B) \end{aligned}$$

we have

$$m([a, b) \cap B) = m([0, 1/n) \cap B) = m(B)/n = m([a, b))m(B).$$

Suppose now that $m([a, b)) = r/n$ with $r, n \in \mathbf{N}$. Then

$$\begin{aligned} m([a, b) \cap B) &= \sum_{k=0}^{r-1} m([a + k/n, a + (k+1)/n) \cap B) \\ &= \sum_{k=0}^{r-1} m([a + k/n, a + (k+1)/n)) \cdot m(B) \\ &= m([a, b))m(B). \end{aligned}$$

For the general case, let $\{r_n\}$ be a sequence of rationals such that $r_n \to m([a, b))$ with $r_n \leq m([a, b))$. Then, by monotone convergence,

$$\begin{aligned} m([a, b) \cap B) &= \lim_{n\to\infty} m([a, a + r_n) \cap B) \\ &= \lim_{n\to\infty} m([a, a + r_n)) \cdot m(B) \\ &= m([a, b))m(B). \end{aligned}$$

Finally, to end the proof of the lemma, let O be any open subset of $[0, 1)$ and let $\{(a_n, b_n)\}$ be a sequence of pairwise disjoint intervals such that $O = \bigcup(a_n, b_n)$. We then have

$$m(O \cap B) = \sum_{n=1}^{\infty} m((a_n, b_n) \cap B) = \sum_{n=1}^{\infty} m((a_n, b_n)) \cdot m(B) = m(O)m(B).$$

We can now establish our main result for the case when T is an arbitrary interval.

2.3 **Theorem.** *If $C, T \subset \mathbf{R}$ with T an interval, then $m^*(\bigcup_{t\in C} E_t \cap T)$ is 0 or mT.*

Proof. In view of the remarks at the beginning of this section, the result will follow if we show that the outer measure of $A = \bigcup_{t\in C} A_t$ equals 0 or 1. For the proof we consider two cases.

Case 1: A is measurable.

Since A_0 is dense in $[0, 1)$ it suffices to prove, in view of Lemma 2.2, that $s \in A_0 \Rightarrow A \oplus s = A$. To show that this is indeed the case, take $s \in A_0$ and let $m, n \in \mathbf{Z}$ be such that $s = m\alpha_1 \oplus n\alpha_2$. Observe first that, for any $t \in \mathbf{R}$, it follows from the definition of A_t that

$$\begin{aligned} x \in A_t \oplus s &\Leftrightarrow x = a_t + s + r \quad \text{for some } a_t \in A_t,\ r \in \mathbf{Z} \\ &\Leftrightarrow x = t + p\alpha_1 + q\alpha_2 + k + m\alpha_1 + n\alpha_2 + r \quad \text{for some } p, q, r, k \in \mathbf{Z} \\ &\Leftrightarrow x = t \oplus n_1\alpha_1 + n_2\alpha_2 \quad \text{for some } n_1, n_2 \in \mathbf{Z} \\ &\Leftrightarrow x \in A_t. \end{aligned}$$

Now, if $B_\alpha \subset [0, 1)$ with α in some family of indexes, we have, for all $x \in [0, 1)$,

$$\begin{aligned} (\mathscr{T}_s \chi_{\bigcup B_\alpha})(x) &= \chi_{\bigcup B_\alpha}(x \ominus s) = \max_\alpha\{\chi_{B_\alpha}(x \ominus s)\} \\ &= \max_\alpha\{\chi_{B_\alpha \oplus s}(x)\} = \chi_{\bigcup(B_\alpha \oplus s)}(x) \\ &= \chi_{\bigcup \mathscr{T}_s B_\alpha}(x). \end{aligned}$$

Thus, $\mathscr{F}_s \bigcup_\alpha B_\alpha = \bigcup_\alpha \mathscr{F}_s B_\alpha$. But then

$$A \oplus s = \mathscr{F}_s A = \mathscr{F}_s \bigcup_{t \in C} A_t = \bigcup_{t \in C} \mathscr{F}_s A_t = \bigcup_{t \in C} A_t \oplus s = \bigcup_{t \in C} A_t = A$$

and the desired result follows.

Case 2: A is nonmeasurable.

Since A is nonmeasurable, $0 < m^* A \leq 1$. We want to show that $m^* A = 1$. Let $\varepsilon > 0$ and let $U \supset A$ be an open set such that $mU \leq m^* A + \varepsilon$.

Observe first that, if $s \in A_0$, then

$$\chi_A(x) = \chi_{A \oplus s}(x) = (\mathscr{F}_s \chi_A)(x) = \chi_A(x \ominus s) \leq \chi_U(x \ominus s) = (\mathscr{F}_s \chi_U)(x)$$

and

$$\mathscr{F}_s U = U \oplus s = ((U \cap [0, 1 - s)) + s) \cup ((U \cap [1 - s, 1)) + (s - 1))$$

implying that $\mathscr{F}_s U$ is a Borel set containing A. Since $A_0 = \bigcup_{r \in \mathbf{Z}} E_r \cap [0, 1)$ is countable, it follows that $W := \bigcap_{s \in A_0} \mathscr{F}_s U$ is also a Borel set that contains A.

Now, if $B_\alpha \subset [0, 1)$ with α in some family of indexes, we have

$$\mathscr{F}_t \bigcap_\alpha B_\alpha = \bigcap_\alpha \mathscr{F}_t B_\alpha$$

for any $t \in \mathbf{R}$, and so

$$\mathscr{F}_t W = \mathscr{F}_t \bigcap_{s \in A_0} \mathscr{F}_s U = \bigcap_{s \in A_0} \mathscr{F}_t \mathscr{F}_s U = \bigcap_{s \in A_0} \mathscr{F}_{t+s} U.$$

Thus, $t \in A_0 \Rightarrow W \oplus t = \mathscr{F}_t W = W$.

Applying Lemma 2.2 to W, we have $mW = 0$ or $mW = 1$. Since $m^* A > 0$ and $W \supset A$, we have $mW = 1$. But then, since $W \subset \mathscr{F}_0 U = U$,

$$m^* A \geq mU - \varepsilon \geq mW - \varepsilon = 1 - \varepsilon$$

and, since ε is arbitrary, $m^* A = 1$. The proof is now complete.

3. The case T measurable

In the previous section we proved that no matter which C we choose, if T is any interval, then $m^*(\bigcup_{t \in C} E_t \cap T)$ is 0 or mT. We shall now prove that this result remains valid if T is any Lebesgue measurable set. To do so, we require the following auxiliary result.

3.1 **Lemma.** *If* $C \subset \mathbf{R}$ *and* $m^*(\bigcup_{t \in C} E_t \cap I_0) = 0$ *for some open interval* $I_0 \subset \mathbf{R}$, *then* $m^*(\bigcup_{t \in C} E_t \cap T) = 0$ *for any* $T \subset \mathbf{R}$.

Proof. Let $C \subset \mathbf{R}$ and set $M := \bigcup_{t \in C} E_t$. Let I_0 be an open interval for which $m^*(M \cap I_0) = 0$, and let $I = (a, b) \subset I_0$ with $a \in E_0$, an interval that can be found since E_0 is dense in $\mathbf{R}$.

Observe first that $s \in E_0 \Rightarrow E_t + s = E_t$ for all $t \in \mathbf{R}$, and so

$$M + s = \left(\bigcup_{t \in C} E_t \right) + s = \bigcup_{t \in C} (E_t + s) = \bigcup_{t \in C} E_t = M \qquad (s \in E_0).$$

It then follows that, given $x \in E_0$ and $y > x$, since $-x \in E_0$, we have

$$\begin{aligned} m^*(M \cap (x, y)) &= m^*((M \cap (x, y)) - x) \\ &= m^*((M - x) \cap ((x, y) - x)) \\ &= m^*(M \cap (0, y - x)). \end{aligned}$$

Now, let $s, t \in \mathbf{R}$ with $s \in E_0$ and $0 < t - s < b - a$. By the above remark, we have

$$m^*(M \cap (s, t)) = m^*(M \cap (0, t - s)) \leq m^*(M \cap (0, b - a)) = m^*(M \cap I) = 0.$$

Suppose now that we are given an open interval (c, d). Let $\varepsilon > 0$, and let $t_1, \dots, t_n \in E_0$ be such that $c < t_1 < \cdots < t_n < d$, $t_{i+1} - t_i < b - a$ $(i = 1, \dots, n-1)$, $t_1 - \varepsilon/2 < c$, and $t_n + \varepsilon/2 > d$. By the last result, it follows that

$$\begin{aligned} &m^*(M \cap (c, d)) \\ &\quad = m^*\left(M \cap \bigcup_{i=1}^{n-1}[t_i, t_{i+1})\right) + m^*(M \cap (c, t_1)) + m^*(M \cap [t_n, d)) \\ &\quad \leq \sum_{i=1}^{n-1} m^*(M \cap (t_i, t_{i+1})) + \varepsilon = \varepsilon \end{aligned}$$

and we conclude that $m^*(M \cap (c, d)) = 0$. Therefore, if O is any open set and $\{O_i\}$ is a countable collection of disjoint open intervals whose union is O, then

$$m^*(M \cap O) = \sum_{i=1}^{\infty} m^*(M \cap O_i) = 0.$$

Finally, if T is an arbitrary subset of $\mathbf{R}$, then $m^*(M \cap T) \leq m^*(M \cap \mathbf{R}) = 0$, as claimed.

3.2 **Theorem.** *If $C, T \subset \mathbf{R}$ with T measurable, $m^*(\bigcup_{t \in C} E_t \cap T)$ is 0 or mT.*

Proof. Let $M := \bigcup_{t \in C} E_t$, and suppose that $m^*(M \cap T) \neq 0$. We are going to prove that $m^*(M \cap T) = mT$.

By Lemma 3.1, the assumption $m^*(M \cap T) \neq 0$ implies that $m^*(M \cap I) \neq 0$ for any open interval I in $\mathbf{R}$. By Theorem 2.3, it follows that $m^*(M \cap I) = mI$ for any open interval I in $\mathbf{R}$. Let O be any open set in $\mathbf{R}$, and let $\{O_i\}$ be a countable collection of disjoint open intervals whose union is O. Then

$$m^*(M \cap O) = \sum_{i=1}^{\infty} m^*(M \cap O_i) = \sum_{i=1}^{\infty} mO_i = mO.$$

Let $\varepsilon > 0$. Since T is measurable, there exists $O \supset T$ open such that $m(O \sim T) < \varepsilon$. In view of the above relation,

$$mT \leq mO = m^*(M \cap O) = m^*(M \cap T) + m^*(M \cap (O \sim T)) \leq m^*(M \cap T) + \varepsilon.$$

This shows that $m^*(M \cap T) = mT$ and the proof of the theorem is complete.

4. An example with T nonmeasurable

In this section we provide an example of (nonmeasurable) sets C and T for which the outer measure of $\bigcup_{t \in C} E_t \cap T$ lies in between 0 and the outer measure of T.

Let α_1, α_2 be any real numbers with α_1/α_2 irrational. As before, let

$$E_t = \{t + m\alpha_1 + n\alpha_2 | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R}).$$

Define a relation " $\sim$ " on $\mathbf{R}$ by letting $x \sim y$ hold if and only if $x - y \in E_0$. This is an equivalence relation and hence partitions $\mathbf{R}$ into equivalence classes, each having the form E_t for some $t \in \mathbf{R}$. By the axiom of choice there exists a set C which contains exactly one element from each equivalence class.

Consider the sets

$$F_t := \{t + m\alpha_1 + 2n\alpha_2 | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R})$$

and

$$G_t := \{t + m\alpha_1 + (2n+1)\alpha_2 | m, n \in \mathbf{Z}\} \qquad (t \in \mathbf{R})$$

and observe that F_t belongs to the family of sets we have been considering, with $\alpha_1/2\alpha_2$ irrational. Let $M := C + F_0 = \bigcup_{t\in C} F_t$ and $T := (C + G_0) \cup [0, 1] = \bigcup_{t\in C} G_t \cup [0, 1]$.

It is well known that $M = C + F_0$ is a nonmeasurable set and, moreover, each Lebesgue measurable set that is included in it or in its complement $M^c = C + G_0 = M + \alpha_2$ has Lebesgue measure zero (see, for example, [1, 2]). Since also $M \cap [0, 1]$ is nonmeasurable it follows, by Theorem 3.2, that

$$m^*(M \cap T) = m^*(M \cap (M^c \cup [0, 1])) = m^*(M \cap [0, 1]) = 1.$$

On the other hand,

$$m^*T = m^*(M^c \cup [0, 1]) \geq m^*M^c = m^*(M + \alpha_2) = \infty$$

showing that $0 < m^*(M \cap T) < m^*T$.

References

1. D. L. Cohn, *Measure theory*, Birkhäuser, Boston, 1980.

2. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

DEPARTAMENTO DE MÉTODOS MATEMÁTICOS Y NUMÉRICOS, IIMAS-UNAM, APARTADO POSTAL 20-726, MÉXICO D.F., 01000, MÉXICO