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THE LIFTING OF THE UKK PROPERTY FROM E TO C_E

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ABSTRACT. In this paper we show that C_E, the unitary matrix space associated with the symmetrically normed sequence space E, has the UKK property for the weak operator topology if E has the UKK property for the pointwise convergence topology. We also prove that the quasi-normed space $C_p = C_{l_p}$, for $0 < p < 1$, has the UKK property for the weak operator topology.

1. INTRODUCTION

Let Φ be a symmetric norm on F, the space of infinite sequences with only finitely many nonzero elements, i.e., Φ is invariant under permutations and depends only on the absolute values of coordinates. The maximal symmetric sequence space associated to Φ, denoted by E_Φ, is defined by

$$E_\Phi = \{x : \lim_{n\to\infty} \Phi(x_1, x_2, \dots, x_n, 0, 0, \dots) < \infty\},$$

with norm $\|x\|_{E_\Phi} = \lim_{n\to\infty} \Phi(x_1, x_2, \dots, x_n, 0, 0, \dots)$. The minimal symmetric sequence space associated to Φ, denoted by $E_\Phi^{(0)}$, is defined to be the closure (in E_Φ) of F.

Let E be a general symmetric sequence space lying between $E_\Phi^{(0)}$ and E_Φ. The *unitary matrix space* C_E associated with E is the Banach space of all compact operators on l_2 for which $\lambda(A) \in E$, normed by $\Phi(A) = \|(\lambda(A))\|_E$. Here $\lambda(A) = (\lambda_n(A))$ is the sequence of s-numbers of A, i.e., the eigenvalues of $(A^*A)^{1/2}$ arranged in a nonincreasing ordering counting multiplicity.

Let $(X, \|\cdot\|)$ be a Banach space, and let τ be a topological vector space topology on X that is weaker than the norm topology.

Definition 1.1. X is said to have the *Kadec-Klee* property with respect to τ, denoted by $KK(\tau)$, if whenever $\{x_n\}_{n=1}^\infty$ is a sequence in X such that $x_n \to x \in X$ with respect to τ and $\|x_n\| \to \|x\|$, then it follows that $\|x_n - x\| \to 0$.

Definition 1.2. X is said to have the *uniform Kadec-Klee property* with respect to τ, denoted by $UKK(\tau)$, if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that

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whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in the unit ball of X such that $x_n \longrightarrow x \in X$ with respect to τ and $\inf_{n\neq m} \|x_n - x_m\| \geq \varepsilon$, then it follows that $\|x\| \leq 1 - \delta(\varepsilon)$.

It is easy to show that the UKK property implies the KK property.

For a symmetric sequence space E and its associated unitary matrix space C_E we are interested in the topology of pointwise convergence and the weak operator topology respectively.

Arazy [1] and Simon [14] show that if E has the KK property with respect to the pointwise convergence topology, then C_E has the KK property with respect to the weak operator topology. Our main result (Theorem 3.1) is the analogous result for the UKK property. As a consequence of the main result, C_E has a fixed point property for nonexpansive mappings if E has the UKK property. We must also mention here the recent interesting results in [2] concerning relations between the UKK property for a symmetric Banach function space E and the corresponding space $E(\mathscr{M})$ of all τ-measurable operators affiliated with a von Neumann algebra $\mathscr{M}$ that supports a faithful, normal, semi-finite trace τ whose decreasing rearrangement lies in E: if E is α-convex with constant 1 for some $0 < \alpha \leq 1$ and if E satisfies a lower-q estimate with constant 1 for some finite $q \geq \alpha$, then $E(\mathscr{M})$ has the UKK property for the topology of local convergence in measure. As a special case of this result it follows that C_p $(0 < p < 1)$ has the UKK property for the weak operator topology. In Section 4 below we present a short elementary proof of this fact.

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2. Preliminaries

The following definition is useful in studying the UKK property.

Definition 2.1. The *$UKK(\tau)$-modulus* of a space X is defined by $\delta_X(\varepsilon) = \inf\{1 - \|x\|\}$, where the infimum is taken over all x such that x is the τ-limit of a sequence $\{y_n\}$ in the unit ball of X with $\inf_{n\neq m} \|y_n - y_m\| \geq \varepsilon$.

Lemma 2.2. *If E has the KK property for the topology of pointwise convergence then E is minimal (i.e., $E = E_\Phi^{(0)}$).*

Proof. Let $x \in E$. Then $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \to x$ pointwise and $\|x^{(n)}\|_E \to \|x\|_E$. By the KK property, $\|x^{(n)} - x\|_E \to 0$; hence $x \in E_\Phi^{(0)}$.

Lemma 2.3. *If E has the UKK property for the topology of pointwise convergence then E is maximal (i.e., $E = E_\Phi$).*

Proof. Suppose that E is not maximal. Then $(e_n)_{n=1}^{\infty}$ is not a boundedly complete Schauder basis (see [10, Definition 1.b.3]). Therefore by [10, Theorem 1.c.10] E contains a subspace isomorphic to c_0. By [7, Lemma 2.2], for any $\varepsilon > 0$, there is a sequence $\{y_n\}$ of elements of the unit ball such that

$$(1-\varepsilon)\sup|a_i| \leq \|\sum a_i y_i\|_E \leq \sup|a_i|$$

for all finite sequences $\{a_n\}$ of real numbers. Let $z_n = y_1 + y_n$; then $\|z_n\|_E \leq 1$ and $\|z_n - z_m\|_E \geq 1 - \varepsilon$. Clearly $y_n \to 0$ pointwise in E, and so $z_n \to y_1$

pointwise in E. Let $\delta(\,\cdot\,)$ denote the UKK-modulus for the topology of pointwise convergence. Then

$$1 - \delta(1-\varepsilon) \geq \|y_1\|_E \geq 1 - \varepsilon,$$

which is a contradiction for $\varepsilon > 0$ sufficiently small.

Example 2.4. For every $x \in c_0$, define $\|x\| = \sum 2^{-n} x_n^*$ where (x_n^*) is the decreasing rearrangement of $(|x_n|)$. Then $(c_0, \|\cdot\|)$ has the KK property but not the UKK property for the topology of pointwise convergence. Since c_0 is not maximal, it follows that Lemma 2.3 breaks down if " UKK " is replaced by " KK ".

Proof. Let $\{x^{(k)}\}$ be a sequence in c_0 converging pointwise to $x \in c_0$ and with $\|x^{(k)}\| \to \|x\|$. Without loss of generality we may assume that $x = (x_n^*)$ and that $\|x^{(k)}\| = \|x\| = 1$. Given $\varepsilon > 0$, there exists N such that $x_i < \varepsilon/2$ for all $i > N$, and so $\|(x_1, x_2, \dots, x_N, 0, 0, \dots)\| \geq 1 - 2^{-(N+1)}\varepsilon$. Since $x^{(k)} \to x$ pointwise, there exists M such that

$$\|(x_1^{(k)}, x_2^{(k)}, \dots, x_N^{(k)}, 0, 0, \dots) - (x_1, x_2, \dots, x_N, 0, 0, \dots)\| < 2^{-(N+1)}\varepsilon$$

for all $k > M$. Thus,

$$\|(x_1^{(k)}, x_2^{(k)}, \dots, x_N^{(k)}, 0, 0, \dots)\| > 1 - 2^{-N}\varepsilon$$

for all $k > M$. By an obvious rearrangement inequality, for all $k > M$ we have

$$\|(x_1^{(k)}, x_2^{(k)}, \dots, x_N^{(k)}, 0, 0, \dots)\| + 2^{-N}\|(x_{N+1}^{(k)}, x_{N+2}^{(k)}, \dots, x_n^{(k)} \dots)\| \leq \|x^{(k)}\| \leq 1.$$

Hence $\|(x_{N+1}^{(k)}, x_{N+2}^{(k)}, \dots, x_n^{(k)}, \dots)\| \leq \varepsilon$ for all $k > M$. Thus,

$$\begin{aligned} \|x^{(k)} - x\| &\leq \|(x_1^{(k)}, x_2^{(k)}, \dots, x_N^{(k)}, 0, 0, \dots) - (x_1, x_2, \dots, x_N, 0, 0, \dots)\| \\ &\quad + \|(x_{N+1}^{(k)}, x_{N+2}^{(k)}, \dots, x_n^{(k)}, \dots)\| + \|(x_{N+1}, x_{N+2}, \dots, x_n, \dots)\| \\ &\leq (2^{-(N+1)} + 1 + 2^{-1})\varepsilon < 2\varepsilon \end{aligned}$$

for all $k > M$. So $x^{(k)} \to x$ in norm. Therefore, c_0 has the KK property with respect to the topology of pointwise convergence. It is interesting to note that by [1,14] this symmetric KK norm on c_0 will lift to a symmetric KK norm on the ideal of compact operators on l_2.

In the following δ_E denotes the UKK-modulus for the topology of pointwise convergence in a symmetric sequence space E. We estimate δ_E in terms of a geometrical quantity β_E.

Proposition 2.5. *Let E be a symmetric sequence space with the KK property, and let*

$$\beta_E(\varepsilon) = \inf\{1 - \|x\|_E : \|x+y\|_E \leq 1,\ x, y \text{ disjoint and } \|y\|_E \geq \varepsilon,\ x, y \in E\}.$$

Then $\delta_E(\varepsilon/2) \leq \beta_E(\varepsilon) \leq \delta_E(2\varepsilon)$.

Proof. Given $\eta > 0$, choose $x, y \in E$ disjoint with $\|x+y\|_E \leq 1$, $\|y\|_E \geq \varepsilon$ such that $1 - \|x\|_E < \beta_E(\varepsilon) + \eta$. Since E is minimal, we can choose N such that

$$\|(x_1, x_2, \dots, x_N, 0, 0, \dots) - x\|_E < \eta$$

and

$$\|(y_1, y_2, \dots, y_N, 0, 0, \dots)\|_E > \varepsilon/2.$$

Define for $k = 1, 2, 3, \dots$,

$$z^{(k)} = (x_1, x_2, \dots, x_N, \underbrace{0, 0, \dots, 0}_{kN \text{ times}}, y_1, y_2, \dots, y_N, 0, 0, \dots).$$

Then

$$\|z^{(k)}\|_E \leq \|x+y\|_E \leq 1, \quad \|z^{(k)} - z^{(j)}\|_E > \varepsilon/2, \quad j \neq k,$$

and $z^{(k)} \to (x_1, x_2, \dots, x_N, 0, 0, \dots)$ in the pointwise convergence topology. Therefore by the definition of the modulus we have

$$\begin{aligned} \delta_E(\varepsilon/2) &\leq 1 - \|(x_1, x_2 \dots, x_N, 0, 0, \dots)\|_E \\ &\leq 1 - \|x\|_E + \eta \\ &\leq \beta_E(\varepsilon) + 2\eta. \end{aligned}$$

Since η is arbitrary, $\delta_E(\varepsilon/2) \leq \beta_E(\varepsilon)$. Given $\eta > 0$, choose x, $\{y^{(n)}\}$ with $\|y^{(n)}\|_E \leq 1$, $\inf_{m \neq n} \|y^{(n)} - y^{(m)}\|_E \geq 2\varepsilon$, and $y^{(n)} \to x$ pointwise, such that $1 - \|x\|_E \leq \delta_E(2\varepsilon) + \eta$. Since $y^{(n)} \to x$ pointwise, there exists $M > 0$ such that for all $m > M$, we have (where N is chosen as before)

$$\|(y_1^{(m)}, y_2^{(m)}, \dots, y_N^{(m)}, 0, 0, \dots) - (x_1, x_2, \dots, x_N, 0, 0, \dots)\|_E < \eta.$$

Therefore, for all $m > M$,

$$\|(y_1^{(m)}, y_2^{(m)}, \dots, y_N^{(m)}, 0, 0, \dots) - x\|_E < 2\eta.$$

Since $\inf_{n \neq m} \|y^{(m)} - y^{(n)}\|_E \geq 2\varepsilon$, there are infinitely many $y^{(m)}$ such that

$$\|y^{(m)} - x\|_E \geq \varepsilon.$$

Hence $\|y^{(m)} - (y_1^{(m)}, \dots, y_N^{(m)}, 0, 0, \dots)\|_E \geq \varepsilon - 2\eta$ for such m. Therefore

$$\begin{aligned} \beta_E(\varepsilon - 2\eta) &\leq 1 - \|(y_1^{(m)}, y_2^{(m)}, \dots, y_N^{(m)}, 0, 0, \dots)\| \\ &\leq 1 - \|x\|_E + 2\eta \\ &\leq \delta_E(2\epsilon) + 3\eta. \end{aligned}$$

Since η is arbitrary, we have $\beta_E(\varepsilon) \leq \delta_E(2\varepsilon)$.

The proof of the following lemma may be found in Simon [14].

Lemma 2.6. *Let Φ be an arbitrary symmetric norm.*

(a) *If P is an orthogonal projection and $Q = I - P$, and if $A \in C_E$, then $PAP + QAQ \in C_E$ and $\Phi(PAP + QAQ) \leq \Phi(A)$.*

(b) *If A^*A and B^*B lie in C_E, then A^*B lies in C_E and*

$$\Phi(A^*B) \leq \Phi(A^*A)^{1/2}\Phi(B^*B)^{1/2}.$$

(c) *If $A_n^{(j)} \to A^{(j)}$, $j = 1, \dots, k$, weakly with $A_n^{(j)}$, $A^{(j)} \in C_E$, then we can find an increasing sequence of finite rank projections P_n with $\lim P_n = I$ in the strong operator topology such that $\Phi(P_n A_n^{(j)} P_n - A^{(j)}) \to 0$, $j = 1, \dots, k$.*

Lemma 2.7. *Let $A \in C_E$. If P, Q are orthogonal projections, then*

$$\Phi(PAQ) \leq \Phi(|A|)^{1/2}\Phi(Q|A|Q)^{1/2}.$$

Proof.

$$\begin{aligned}\Phi(PAQ) &\leq \Phi(AQ)\\ &= \Phi(W|A|Q) \text{ (for some partial isometry } W)\\ &\leq \Phi(|A|Q) = \Phi(|A|^{1/2}|A|^{1/2}Q)\\ &\leq \Phi(|A|)^{1/2}\Phi(Q|A|Q)^{1/2} \text{ (by lemma 2.6(b))}.\end{aligned}$$

For a proof of the following lemma the reader is referred to [5, Theorem 5.1, Lemma 5.2].

Lemma 2.8. *Let Φ be a symmetric norm, and let $E = E_\Phi$.*

(a) *C_E coincides elementwise with C_∞ the space of all compact operators if and only if $\lim_{n\to\infty} \Phi(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots) < \infty$.*

(b) *Suppose C_E does not coincide elementwise with C_∞. Whenever a bounded operator A is the weak limit of a sequence of operators $\{A_m\}_1^\infty$ from C_E such that $\sup_m \Phi(A_m) < \infty$, then A also belongs to C_E, and $\Phi(A) \leq \sup_m \Phi(A_m)$.*

Proposition 2.9. *If a symmetric space (E, Φ) has the UKK property for the topology of pointwise convergence, then the closed unit ball of C_E is sequentially compact for the weak operator topology.*

Proof. Let $\langle A_n\rangle$ be a sequence in the closed unit ball of C_E and let $\{\varphi_i\}$ be an orthonormal basis for the underlying Hilbert space H. Then for fixed i, j each sequence $\langle(A_n\varphi_i, \varphi_j)\rangle_{n=1}^\infty$ lies in the closed inteval $[-1, 1]$, since

$$\begin{aligned}|(A_n\varphi_i, \varphi_j)| &\leq \|A_n\varphi_i\| \cdot \|\varphi_j\| \leq \|A_n\| \cdot \|\varphi_i\| \cdot \|\varphi_j\|\\ &\leq \|A_n\| \leq \Phi(A_n) \leq 1.\end{aligned}$$

By a diagonal process there is a subsequence $\langle A_{n_k}\rangle$ and a bounded operator A such that $(A_{n_k}\varphi_i, \varphi_j) \to (A\varphi_i, \varphi_j)$ for all i, j; in particular, $A_{n_k} \to A$ in the weak operator topology. Suppose, to derive a contradiction, that $\sup_n \Phi(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots) = \infty$. Since Φ has the UKK property, E is maximal by Lemma 2.3. Thus $x = (1, 1, \dots, 1, \dots) \in E$, and so $x_k = (\underbrace{1, 1, \dots, 1}_{k}, 0, 0, \dots)$ converges to x pointwise and $\Phi(x_k) \to \Phi(x)$, which implies that $\Phi(x_k - x) \to 0$ by the KK property for E; but $\Phi(x_k - x) = \Phi(\underbrace{0, 0, \dots, 0}_{k}, 1, 1, \dots) \geq 1$ for all k, which is the desired contradiction. Hence, by Lemma 2.8(a), C_E and C_∞ do not coincide elementwise, and by Lemma 2.8(b), A is in the closed unit ball of C_E.

Remark 2.10. Example 2.4 shows that Proposition 2.9 breaks down if " UKK " is replaced by " KK ".

3. UKK PROPERTY FOR E IMPLIES UKK PROPERTY FOR C_E

Theorem 3.1. *If a symmetric sequence space E with norm Φ has the UKK property for the pointwise convergence topology, then C_E has the UKK property* *for the weak operator topology. Moreover, if δ_E and δ_{C_E} denote the corresponding UKK-moduli for E and C_E respectively, then $\delta_{C_E}(\varepsilon) \geq \frac{1}{2}\delta_E(\varepsilon^2/128)$.*

Proof. Suppose $\langle A_n \rangle$ is a sequence in the the unit ball of C_E and that $A_n \to A$ in the weak operator topology and $\inf_{n \neq m} \Phi(A_m - A_n) \geq \varepsilon > 0$. Without loss of generality we may assume $|A_n| \xrightarrow{w} B$, $|A_n^*| \xrightarrow{w} C$ for some B and C belong to the closed unit ball of C_E, since by Proposition 2.9 the unit ball of C_E is sequentially compact. By Lemma 2.6(c) there is a sequence P_n of finite rank orthogonal projections such that $P_n \uparrow I$ strongly, $\Phi(P_n A_n P_n - A) \to 0$, $\Phi(P_n |A_n| P_n - B) \to 0$, and $\Phi(P_n |A_n^*| P_n - C) \to 0$. Let $Q_n = I - P_n$. Since there are infinitely many A_n with $\Phi(A_n - A) \geq \varepsilon/2$, we may assume, by passing to a subsequence, that

$$\frac{\varepsilon}{2} \leq \Phi(A_n - A) \leq \Phi(P_n A_n P_n - A) + \Phi(P_n A_n Q_n) + \Phi(Q_n A_n Q_n) + \Phi(Q_n A_n P_n)$$

for all n. Since $\Phi(P_n A_n P_n - A) \to 0$, one of the following must hold:

(i) $\Phi(Q_n A_n Q_n) \geq \varepsilon/8$ for infinitely many n;
(ii) $\Phi(P_n A_n Q_n) \geq \varepsilon/8$ for infinitely many n;
(iii) $\Phi(Q_n A_n P_n) \geq \varepsilon/8$ for infinitely many n.

By passing to a subsequence, we may suppose that one of the three cases holds for all n.

Case 1. Suppose (i) holds for all n. Define $x^{(1)}$ to be the sequence obtained by first listing all the singular numbers of $P_1 A_1 P_1$, including enough zeros to have $\dim P_1$ entries, and then after that listing finitely many singular numbers of $Q_1 A_1 Q_1$ so that $\Phi(\lambda_1(Q_1 A_1 Q_1), \ldots, \lambda_{s_1}(Q_1 A_1 Q_1)) \geq \varepsilon/16$, and finally zeros after that.

Now suppose that $x^{(j-1)}$ has been defined. We define x^j inductively by first listing all the singular numbers of $P_{n_j} A_{n_j} P_{n_j}$, where $n_j > n_{j-1}$ is chosen such that n_j is the least number such that $\dim P_{n_j} >$ the length of $x^{(j-1)}$ (here the length of $x^{(j)}$ is defined to equal $\min\{n : x_i^{(j)} = 0 \ \forall i > n\}$), and then listing finitely many singular number of $Q_{n_j} A_{n_j} Q_{n_j}$ such that

$$\Phi(\lambda_1(Q_{n_j} A_{n_j} Q_{n_j}), \lambda_2(Q_{n_j} A_{n_j} Q_{n_j}), \ldots, \lambda_{s_j}(Q_{n_j} A_{n_j} Q_{n_j})) \geq \varepsilon/16.$$

By Lemma 2.6(a), we have

$$\Phi(x^{(j)}) \leq \Phi(P_{n_j} A_{n_j} P_{n_j} + Q_{n_j} A_{n_j} Q_{n_j}) \leq \Phi(A_{n_j}) \leq 1.$$

Thus we have defined a sequence $\langle x^{(n)} \rangle$ in the unit ball of E. Let x be the sequence of singular numbers of A. Now $\|P_n A_n P_n - A\| \to 0$, and so $x_i^{(j)} \to x_i$; also, for $j > i$,

$$\Phi(x^{(i)} - x^{(j)}) \geq \varepsilon/16.$$

Since E has the UKK property, we have $\Phi(x) \leq 1 - \delta_E(\varepsilon/16)$. So $\Phi(A) = \Phi(x) \leq 1 - \delta_E(\varepsilon/16)$.

Case 2. Suppose (ii) holds for all n. By Lemma 2.7

$$\Phi(P_n A_n Q_n) \leq \Phi(|A_n|)^{1/2} \Phi(Q_n |A_n| Q_n)^{1/2},$$

i.e. $\Phi(Q_n |A_n| Q_n) \geq \Phi(P_n A_n Q_n)^2 \geq \varepsilon^2/64$.

Define $x^{(j)}$ in the same way as in Case 1, changing A_{n_j} to $|A_{n_j}|$, A to B, and $\varepsilon/8$ to $\varepsilon^2/64$. We obtain $\Phi(B) = \Phi(x) \leq 1 - \delta_E(\varepsilon^2/128)$. But by Lemma 2.7, $\Phi(P_n A_n P_n)^2 \leq \Phi(P_n|A_n|P_n)$, and we have

$$\Phi(A) = \lim_{j\to\infty} \Phi(P_{n_j} A_{n_j} P_{n_j}) \leq \lim_{j\to\infty} \Phi(P_{n_j}|A_{n_j}|P_{n_j})^{1/2} = \Phi(B)^{1/2}.$$

So $\Phi(A) \leq \Phi(B)^{1/2} \leq \sqrt{1 - \delta_E(\varepsilon^2/128)} \leq 1 - \frac{1}{2}\delta_E(\varepsilon^2/128)$.

Case 3. Suppose (iii) holds for all n. Since $\Phi(Q_n A_n P_n) = \Phi(P_n A_n^* Q_n)$, Lemma 2.7 again gives

$$\Phi(Q_n|A_n^*|Q_n) \geq \Phi(Q_n A_n P_n)^2 \geq \varepsilon^2/64.$$

Similarly we will get

$$\begin{aligned}\Phi(A) &= \lim_{j\to\infty} \Phi(P_{n_j} A_{n_j} P_{n_j}) \\ &\leq \lim_{j\to\infty} \Phi(P_{n_j}|A_{n_j}^*|P_{n_j})^{1/2} \\ &= \Phi(C) \leq 1 - \tfrac{1}{2}\delta_E(\varepsilon^2/128).\end{aligned}$$

Hence C_E has the UKK property for the weak operator topology and $\delta_{C_E}(\varepsilon) \geq \frac{1}{2}\delta_E(\varepsilon^2/128)$.

Remark. It is clear that $\beta_{l_1} = \varepsilon$. By Proposition 2.5, $\delta_{l_1}(\varepsilon) \geq \varepsilon/2$ (in fact $\delta_{l_1}(\varepsilon) = \varepsilon/2$).

So

$$\delta_{C_1}(\varepsilon) \geq \frac{1}{2}\delta_{l_1}(\varepsilon^2/128) \geq \frac{\varepsilon^2}{512}.$$

In Lennard [8] there is the better estimate $1-\delta_{C_1}(\varepsilon) \leq \left(1-(\varepsilon/2\sqrt{3})^2\right)^{1/2}$, which implies $\delta_{C_1}(\varepsilon) \geq \varepsilon^2/24$.

Let K be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$. A map $T : K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. K is said to have the *fixed point property* if it has a fixed point for every nonexpansive mapping. By van Dulst and Sims [3], we have the following corollary.

Corollary 3.2. *If E has the UKK property for the topology of pointwise convergence, then every convex subset of C_E which is compact in the weak operator topology has the fixed point property.*

Remark 3.3. It follows from Proposition 2.9 and the above corollary that if E has the UKK property, then the closed unit ball of C_E has the fixed point property.

4. The UKK property in C_p for $0 < p < 1$

First we give an example to show that Lemma 2.6(a) breaks down for quasi-norms. Thus the proof of Theorem 3.1 appears to break down completely for quasi-norms.

Example 4.1. Let $p = \frac{1}{2}$,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $A = |A|$ and P is an orthogonal projection. Let $Q = I - P$, so that

$$PAP + QAQ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $\|PAP + QAQ\|_p = (1^{1/2} + 1^{1/2})^2 = 4$. But $\|A\|_p = (0^{1/2} + 2^{1/2})^2 = 2$. So we have $\|PAP + QAQ\|_p > \|A\|_p$.

Proposition 4.2. *Let $0 < p \le 2$, let P be an orthogonal projection, and let $Q = I - P$. Then $\|A\|_p^2 \ge \|PA\|_p^2 + \|QA\|_p^2$, and $\|A\|_p^2 \ge \|AP\|_p^2 + \|AQ\|_p^2$ for all $A \in C_p$.*

Proof. By [6, Theorem 4], $\|A\|_p = \min\left(\sum \|Ae_n\|^p\right)^{1/p}$, where the minimum is taken over all orthonomal bases (e_n). Let (e_n) be an orthonormal basis such that $\|A\|_p = \left(\sum \|Ae_n\|^p\right)^{1/p}$. Let $f_n = PAe_n$ and $g_n = QAe_n$; then

$$\begin{aligned}
\|A\|_p &= (\sum \|Ae_n\|^p)^{1/p} = \left(\sum \|(P+Q)Ae_n\|^p\right)^{1/p} \\
&= (\sum \|PAe_n + QAe_n\|^p)^{1/p} = \left(\sum (\|f_n\|^2 + \|g_n\|^2)^{p/2}\right)^{1/p} \\
&= [(\sum (\|f_n\|^2 + \|g_n\|^2)^{p/2})^{2/p}]^{1/2} \\
&\ge [\|\langle \|f_n\|^2 \rangle\|_{p/2} + \|\langle \|g_n\|^2 \rangle\|_{p/2}]^{1/2} \quad \text{(by the reverse Hölder inequality),} \\
&= [(\sum \|f_n\|^p)^{2/p} + (\sum \|g_n\|^p)^{2/p}]^{1/2} \\
&\ge [\|PA\|_p^2 + \|QA\|_p^2]^{1/2}.
\end{aligned}$$

Consequently, $\|A\|_p^2 = \|A^*\|_p^2 \ge \|PA^*\|_p^2 + \|QA^*\|_p^2 = \|AP\|_p^2 + \|AQ\|_p^2$.

Remark. Applying Proposition 4.1 twice we obtain

$$\|A\|_p^2 \ge \|PAP\|_p^2 + \|QAP\|_p^2 + \|PAQ\|_p^2 + \|QAQ\|_p^2.$$

This inequality appears for the case $p = 1$ in [1] and [8].

Theorem 4.3. *The quasi-normed operator ideal C_p $(0 < p < 1)$ has the UKK property with respect to the weak operator topology with $\delta_{C_p}(\varepsilon) \ge (2^{3-6/p}/3)\varepsilon^2$.*

Proof. Let $\langle A_n \rangle$ be a sequence in the unit ball of C_p such that $\|A_m - A_n\|_p \ge \varepsilon$ ($m \ne n$), and such that $A_n \to A$ in the weak operator topology. Let $\{P_n\}_{n=1}^{\infty}$ be finite rank orthogonal projections so that $P_n \uparrow I$ strongly and $\|P_n A_n P_n - A\|_p \to 0$.

Since $\|A_n - A_m\|_p \ge \varepsilon$, $m \ne n$, there is a subsequence $\langle A_{n_k} \rangle$ with

$$\|A_{n_k} - A\|_p \ge \frac{\varepsilon}{2} \cdot 2^{1-\frac{1}{p}} = 2^{\frac{-1}{p}} \cdot \varepsilon.$$

Hence by the remark following Proposition 4.1,

$$\begin{aligned}
2^{\frac{-1}{p}} \cdot \varepsilon &\le \|A_{n_k} - A\|_p \\
&\le 2^{\frac{2}{p}-2}\{\|P_{n_k} A_{n_k} P_{n_k} - A\|_p + \|P_{n_k} A_{n_k} Q_{n_k}\|_p \\
&\qquad + \|Q_{n_k} A_{n_k} P_{n_k}\|_p + \|Q_{n_k} A_{n_k} Q_{n_k}\|_p\} \\
&\le 2^{\frac{2}{p}-2}\{\|P_{n_k} A_{n_k} P_{n_k} - A\|_p + \sqrt{3}(\|A_{n_k}\|_p^2 - \|P_{n_k} A_{n_k} P_{n_k}\|_p^2)^{1/2}\}.
\end{aligned}$$

Taking the limit, we have

$$2^{\frac{-1}{p}} \cdot \varepsilon \le \sqrt{3} \cdot 2^{\frac{2}{p}-2}(1 - \|A\|_p^2)^{1/2}.$$

Thus

$$2^{\frac{-2}{p}} \cdot \varepsilon^2 \leq 3 \cdot 2^{\frac{4}{p}-4}(1 - \|A\|_p^2).$$

So

$$\|A\|_p \leq (1 - \frac{2^{4-\frac{6}{p}}}{3}\varepsilon^2)^{1/2} \leq 1 - \frac{2^{3-\frac{6}{p}}}{3}\varepsilon^2.$$

Therefore, C_p has the UKK property for the weak operator topology with

$$\delta_{C_p}(\varepsilon) \geq \frac{2^{3-6/p}}{3}\varepsilon^2.$$

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