PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

GENERIC EMBEDDINGS AND THE FAILURE OF BOX

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(Communicated by Andreas R. Blass)

ABSTRACT. We prove that if $\{a \subseteq \kappa^+ |$ order type of a is a cardinal$\}$ is stationary, then Jensen's principle $\square_\kappa$ fails. We also show that $\forall\kappa\square_\kappa$ is consistent with a superstrong cardinal.

1. INTRODUCTION

The results of this paper were motivated by the question "Where can ω_1 be sent in Woodin's non-stationary tower?" ([W88]). A set $\mathscr{A} \subset \mathscr{P}(X)$ is *stationary* (in $\mathscr{P}(X)$) iff $\forall f\colon X^{<\omega} \to X \ \exists a \in \mathscr{A} (a \neq X)$ such that a is closed under f. By "ω_1 can be sent to κ" (in symbols $\omega_1 \to \kappa$) we mean $\{a \subseteq \kappa | \operatorname{ot}(a) = \omega_1\}$ is stationary. More generally, a cardinal κ is preserved ($\Pr(\kappa)$) iff $\{a \subseteq \kappa | \operatorname{ot}(a)$ is a cardinal$\}$ is stationary. If κ is Ramsey, then $\omega_1 \to \kappa$; Chang's conjecture is equivalent to $\omega_1 \to \omega_2$ ([KM]). We show below that $\Pr(\kappa^+)$ implies $\neg\square_\kappa$. (It was known that Chang's conjecture implies $\neg\square_{\omega_1}$.) We also show that $\forall\kappa\neg\Pr(\kappa^+)$ is consistent with a superstrong cardinal (by showing that $\forall\kappa\square_\kappa$ is consistent with a superstrong cardinal). It is easy to see that if κ is supercompact, then $\Pr(\kappa^+)$.

We start with some basic well-known definitions and results. A *generic embedding* is an elementary embedding $j\colon V \to (M, E)$ defined in some generic extension of V. We assume that the wellfounded part of (M, E) is collapsed to a transitive set. If P is the partial order of stationary subsets of $\mathscr{P}(\lambda)$ (ordered by inclusion) and $G \subseteq P$ is generic, then we get a generic embedding $j\colon V \to (M, E)$. The model (M, E) is the ultrapower $V^{\mathscr{P}(\lambda)}/G$ and $\operatorname{cp}(j) \leq \lambda$. Also, there is an $A \in M$ (A is [id]) such that $\{B \in M | M \models B \in A\} = j''\lambda$ (we abbreviate this by $j''\lambda \,\overline{\in}\, M$) (see [F] for proofs).

Lemma 1.1. *Assume $j\colon V \to (M, E)$ is a generic embedding with $\operatorname{cp}(j) \leq \lambda$ and $j''\lambda \,\overline{\in}\, M$. Then λ is in the wellfounded part of M, for all $X \subseteq \lambda$ (with $X \in V$) $X \in M$, and $\exists \bar{j} \in M$ such that $\forall \alpha \in \lambda M \models$ "$j(\alpha) = \bar{j}(\alpha)$".*

Proof. This follows easily since $j''\lambda \,\overline{\in}\, M$. $\square$

Definition 1.2. $\Pr(\kappa)$ means that $\{a \subset \kappa | \operatorname{ot}(a)$ is a cardinal$\}$ is stationary.

Lemma 1.3. *The following are equivalent*:

(1) $\Pr(\kappa^+)$.

Received by the editors September 18, 1993 and, in revised form, January 31, 1994.
1991 *Mathematics Subject Classification.* Primary 03E35, 03E55.

(2) *There is a generic embedding* $j\colon V \to (M, E)$ *such that* $\mathrm{cp}(j) \le \kappa^+$, $j''\kappa^+ \,\overline{\in}\, M$, *and* $M \models$ "κ^+ *is a cardinal*".

Proof. (1) $\to$ (2). Force with the stationary subsets of $\mathscr{P}(\kappa^+)$ below $\{a \subset \kappa^+ \mid \mathrm{ot}(a)$ is a cardinal$\}$. So we just need to check that $M \models$ "κ^+ is cardinal". But $M \models$ "$\mathrm{ot}([\mathrm{id}])$ is a cardinal", and $\mathrm{ot}([\mathrm{id}])$ is κ^+.

(2) $\to$ (1). Let $f\colon (\kappa^+)^{<\omega} \to \kappa^+$ (with $f \in V$). In M, $j''\kappa^+$ is closed under $j(f)$ and $\mathrm{ot}(j''\kappa^+) = \kappa^+$ is a cardinal. $\square$

Clearly, $\omega_1 \to \kappa$ implies $\Pr(\kappa)$. A cardinal κ is *Jónsson* iff $\{a \subseteq \kappa \mid |a| = \kappa\}$ is stationary. If C is a set of ordinals, then C' is all limit points of C. $\square_\kappa$ means $\exists\langle C_\alpha\colon \alpha \in \kappa^+\rangle$ such that:

(1) C_α is club in α,
(2) $\beta \in (C_\alpha)'$ implies $C_\beta = C_\alpha \cap \beta$,
(3) $\mathrm{cf}(\alpha) < \kappa$ implies $\mathrm{ot}(C_\alpha) < \kappa$.

A cardinal κ is *superstrong* if there is an elementary embedding $j\colon V \to M$ with $\mathrm{cp}(j) = \kappa$ and $V_{j(\kappa)} \subset M$. If $\mathbb{P}$ is a partial order, then $\mathbb{P}$ is α (α an ordinal) strategically closed iff

$$(\forall p_1 \exists p_2 \dots Q p_\beta \dots)(p_1 \ge p_2 \ge \dots \ge p_\beta \ge \dots)$$

where Q is $\exists$ if β is even and $\forall$ if β is odd and the string of α many quantifiers is interpreted as a game. If M, N are models of ZFC, then $M \sim_\lambda N$ means that $V_\lambda^M = V_\lambda^N$. For basic results about extenders see [MS]. For any other unexplained notions see [J].

2. Failure of box

Theorem 2.1. *Let* κ *be a cardinal. If* $\Pr(\kappa^+)$, *then* $\neg\square_\kappa$.

Proof. Assume that $\{a \subset \kappa^+ \mid \mathrm{ot}(a)$ is a card$\}$ is stationary and therefore there is a generic embedding $j\colon V \to (M, E)$ such that $\mathrm{cp}(j) \le \kappa^+$, $j''\kappa^+ \,\overline{\in}\, M$, and $M \models$ "κ^+ is a card". Everywhere below κ^+ denotes the successor cardinal of κ in V. We assume the wellfounded part of M is collapsed to a transitive set, so by Lemma 1.1, $\kappa^+ \in M$, there is a $\overline{j} \in M$ such that $M \models$ "$\overline{j}\colon \kappa^+ \to \mathrm{Ord}$" and $\forall \alpha \in \kappa^+ M \models$ "$\overline{j}(\alpha) = j(\alpha)$", and for all $X \subseteq \kappa^+$ (with $X \in V$) $X \in M$. Therefore $M \models$ "κ^+ is a successor card" and $\mathrm{cp}(j) < \kappa^+$.

We may assume $j(\kappa^+) > \kappa^+$. (If $j(\kappa^+) = \kappa^+$, then κ^+ is Jónsson and so every stationary subset of κ^+ reflects ([T]) and therefore $\neg\square_\kappa$.)

Towards a contradiction assume $\langle C_\alpha\colon \alpha \in \kappa^+\rangle$ is a $\square_\kappa$ sequence. Now work in M. So $\langle j(C)_\alpha\colon \alpha \in j(\kappa^+)\rangle$ is a $\square_{j(\kappa)}$ sequence. Let $\gamma = \sup_{\alpha \in \kappa^+} \overline{j}(\alpha)$. So $j(\kappa) < \gamma < j(\kappa^+)$ and $\mathrm{cf}(\gamma) = \kappa^+$. Let $\overline{\gamma} = \mathrm{ot}(j(C)_\gamma)$. So $\overline{\gamma} \le j(\kappa)$ and $\mathrm{cf}(\overline{\gamma}) = \kappa^+$.

Since $\overline{\gamma} < \gamma$ and $\mathrm{cf}(\overline{\gamma}) = \kappa^+$, the range of $\overline{j}$ is bounded in $\overline{\gamma}$, say α_b is the bound. Choose $\nu \in (j(C)_\gamma)' \cap (\overline{j}''\kappa^+)'$ with $\mathrm{ot}(j(C)_\gamma \cap \nu) > \alpha_b$. Let η be minimal such that $\overline{j}(\eta) \ge \nu$ (so $\sup \overline{j}''\eta = \nu$). Since $j(C)_\nu = j(C)_\gamma \cap \nu$, we have that $\alpha_b < \mathrm{ot}(j(C)_\nu) < \overline{\gamma}$. If $\overline{j}(\eta) = \nu$, then $j(C)_\nu = j(C_\eta)$ and so $\mathrm{ot}(j(C)_\nu) \in$ range of j, contradiction. So $\overline{j}(\eta) > \nu$. Since $\sup \overline{j}''\eta = \nu$, we have $j(C_\eta) \cap \nu = j(C)_\nu$. Choose ρ such that $\rho < \eta$ and $\mathrm{ot}(j(C)_\nu \cap \overline{j}(\rho)) > \alpha_b$. Since $j(C_\eta) \cap \nu = j(C)_\nu$ and $\overline{j}(\rho) < \nu$, $\mathrm{ot}(j(C_\eta) \cap \overline{j}(\rho)) = \mathrm{ot}(j(C)_\nu \cap \overline{j}(\rho))$. But $\mathrm{ot}(j(C_\eta) \cap \overline{j}(\rho)) = j(\mathrm{ot}(C_\eta \cap \rho))$, a contradiction. $\square$

In [LMS] they show it is consistent (from a 2-huge) that $\omega_1 \to \omega_{\omega+1}$. The above theorem and work of Steel, Mitchell and Schimmerling ([MS], [S], [MSS]) show that $\omega_1 \to \omega_{\omega+1}$ (or just $\Pr(\omega_{\omega+1})$) gives an inner model with a cardinal κ such that $o(\kappa) = \kappa^{++}$. (It is shown in [S] that $\neg\Box_{\aleph_\omega}$ gives such a model.) Schimmerling also gets an inner model of a Woodin cardinal from the failure of weaker principles; this suggests the following questions. Does $\omega_1 \to \omega_{\omega+1}$ imply the failure of the weak square property ($\neg\Box^*_{\aleph_\omega}$)? Does $\omega_1 \to \omega_{\omega+1}$ imply stationary reflection at $\omega_{\omega+1}$?

3. Box and superstrong cardinals

Theorem 3.1. *There is a class forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models$ "ZFC plus $\forall \operatorname{card} \kappa$ $\Box_\kappa$". If $V \models$ "δ is superstrong", then $V^{\mathbb{P}} \models$ "δ is superstrong".*

Proof. $\mathbb{P}$ will be the Easton support iteration of the standard forcings for adding a square sequence. The main point is to check that superstrong cardinals are preserved. For any cardinal κ let $\mathbb{B}_\kappa =$ the set of all functions p such that:

(1) $\operatorname{dom}(p) =$ the limit ordinals $\leq \gamma$ for some $\gamma \in \kappa^+$,
(2) $p(\alpha) \subset \alpha$ is club,
(3) $\operatorname{cf}(\alpha) < \kappa \Rightarrow \operatorname{ot}(p(\alpha)) < \kappa$,
(4) if $\beta \in p(\alpha)'$, then $p(\beta) = p(\alpha) \cap \beta$.

It is easy to check that $\mathbb{B}_\kappa$ is $\kappa + 1$ strategically closed (and so adds no new κ sequences of ordinals) and that $V^{\mathbb{B}_\kappa} \models \Box_\kappa$ (see [J], p. 255).

Given any ordinal β define an iteration $\mathbb{P}(\beta)$ with Easton support (direct limits at inaccessible cardinals, inverse limits everywhere else) by letting $\mathbb{Q}_\alpha$ name $\{1\}$ if $\alpha < \beta$ or if $V^{\mathbb{P}(\beta)\restriction\alpha} \models$ "α is not a cardinal". Otherwise, $\mathbb{Q}_\alpha$ names $\mathbb{B}_\alpha$ (in $V^{\mathbb{P}(\beta)\restriction\alpha}$). The forcing $\mathbb{P}$ we use is $\mathbb{P}(\omega_1)$. The basic factor lemma (see [B], 5.1–5.4) gives that for any γ, $\mathbb{P}(\beta) \cong \mathbb{P}(\beta) \restriction \gamma * \dot{\mathbb{P}}(\gamma)$ (where $\dot{\mathbb{P}}(\gamma)$ names $\mathbb{P}(\gamma)$ in $V^{\mathbb{P}(\beta)\restriction\gamma}$). Also, for any Mahlo cardinal γ, $\mathbb{P}(\beta) \restriction \gamma$ has the γ-cc ([B], 2.4).

Claim. For any γ, $\mathbb{P}(\gamma)$ adds no new γ sequences.

Proof of Claim. We will show that $\forall\alpha$ $\mathbb{P}(\gamma) \restriction \alpha$ is $\gamma+1$ strategically closed, and so the claim follows. We inductively define winning strategies τ_α for $\mathbb{P}(\gamma) \restriction \alpha$ such that:

(1) If $p_0, p_1, \dots$ is any play according to τ_α and $\beta < \alpha$, then $p_0 \restriction \beta$, $p_1 \restriction \beta, \dots$ is according to τ_β.
(2) If $p_0, p_1, \dots$ is any partial play according to τ_α and for some $\beta < \alpha$ and for all i $p_i = p_i \restriction \beta^\frown\langle 1, \dots, \rangle$, then $\tau_\alpha(p_0, \dots) = \tau_\alpha(p_o, \dots) \restriction \beta^\frown\langle 1, \dots\rangle$.

The construction at limit stages is easy (note that if we do not have an inverse limit at α, then $\operatorname{cf}(\alpha) = \alpha > \gamma$). For successor stages assume we have τ_α and let $\dot{\sigma}_\alpha$ be a name such that

$$\Vdash_{\mathbb{P}(\gamma)\restriction\alpha} \text{"}\dot{\sigma}_\alpha \text{ witnesses } \dot{\mathbb{Q}}_\alpha \text{ is } \gamma+1 \text{ strategically closed"}.$$

(We may assume if I plays only 1's, then II's response with $\dot{\sigma}_\alpha$ is 1.) Now let $\tau_{\alpha+1}((p_0, q_0), \dots) = (\tau_\alpha(p_0, \dots), \dot{\sigma}_\alpha(q_0, \dots))$. This completes the proof of the claim. □

It is easy to see that $V^{\mathbb{P}} \models \mathrm{ZFC}$ (see [J], p. 196). Also $V^{\mathbb{P}} \models$ "$\forall \operatorname{card} \kappa$ $\Box_\kappa$": Let G be generic for $\mathbb{P}$. Assume $V[G] \models$ "κ is a card ". So $V[G \restriction \kappa] \models$ "κ is a card". Hence $\mathbb{Q}_\kappa$ names $\mathbb{B}_\kappa$ in $V[G \restriction \kappa]$. Let κ^+ be the successor of κ in $V[G \restriction \kappa]$. So

$$V[G \restriction \kappa + 1] \models \text{“}\kappa \text{ and } \kappa^+ \text{ are still cardinals and } \square_\kappa\text{”}.$$

Since in $V[G \restriction \kappa + 1]$, $\mathbb{P}(\kappa + 1) \cong \mathbb{P}(\kappa^+)$, $\mathbb{P}(\kappa + 1)$ adds no new κ^+ sequences to $V[G \restriction \kappa + 1]$. Hence $v[G] \models$ "$\kappa \,\&\, \kappa^+$ are cardinals and $\square_\kappa$ ".

Finally, suppose $V \models \kappa$ is superstrong. Let $j: V \to M$ witness this (so $V_{j(\kappa)} \subset M$). Let $G \subset \mathbb{P}$ be generic. So $V[G] = V[G_1][H_1][H_2]$ where $G_1 \subset \mathbb{P} \restriction \kappa$, $H_1 \subset \mathbb{P}(\kappa) \restriction j(\kappa)$ and $H_2 \subset \mathbb{P}(j(\kappa))$ come from G. Let $\widetilde{H}_1 = \{p \in H_1 | p$ ends in a tail of 1's$\}$.

Claim. $G_1 * \widetilde{H}_1$ is generic for $j(\mathbb{P} \restriction \kappa)$ over M.

Proof of Claim. Suppose $D \subset j(\mathbb{P} \restriction \kappa)$ is dense and in M. Since $M \models$ "$j(\kappa)$ is Mahlo" (it is superstrong), there is an inaccessible $\lambda < j(\kappa)$ such that $\{p \in \mathbb{P} \restriction \lambda | p^\frown\langle 1, \dots \rangle \in D\}$ is dense in $\mathbb{P} \restriction \lambda$. (Note that $(\mathbb{P} \restriction \lambda)^M = \mathbb{P} \restriction \lambda$.) So there is a $p \in G \restriction \lambda$ such that $p^\frown\langle 1, \dots \rangle \in D$. Hence $p^\frown\langle 1, \dots \rangle \in D \cap (G_1 * \widetilde{H}_1)$. This completes the proof of the claim. □

Hence we can extend j to an elementary embedding $\tilde{j}: V[G_1] \to M[G_1][\widetilde{H}_1]$. Now let E be the $(\kappa, j(\kappa))$ extender derived from $\tilde{j}$. So we get the following commutative diagram:

$$\begin{array}{ccc} V[G_1] & \xrightarrow{\ \tilde{j}\ } & M[G_1][\widetilde{H}_1] \\ & {}_{i_E}\searrow \quad\quad \nearrow_{k} & \\ & \mathrm{Ult}(V[G_1], E) & \end{array}$$

Note that $i_E(\kappa) = j(\kappa)$, $\mathrm{cp}(k) > j(\kappa)$ and $M[G_1][\widetilde{H}_1] \sim_{j(\kappa)} \mathrm{Ult}(V[G_1], E)$. Since $V[G]$ adds no new κ sequences (of ordinals) to $V[G_1]$, $\mathrm{Ult}(V[G], E)$ makes sense and is well founded. Note that $V[G_1] \models \kappa$ is strongly inaccessible and so $V[G] \sim_{\kappa+1} V[G_1]$ and therefore $\mathrm{Ult}(V[G], E) \sim_{j(\kappa)} \mathrm{Ult}(V[G_1], E)$ (and $i_E^{V[G]}(\kappa) = j(\kappa)$ also). We now show that $V[G] \sim_{j(\kappa)} M[G_1][\widetilde{H}_1]$ and so $V[G] \models \kappa$ is superstrong. Clearly $M[G_1][\widetilde{H}_1]_{j(\kappa)} \subset V[G]$. So suppose $x \in (V[G])_{j(\kappa)}$. We may assume that $x \subset \lambda$ where $\lambda < j(\kappa)$ is a cardinal ($j(\kappa)$ is a strong limit cardinal in $V[G]$ since it is superstrong in M and therefore a limit of Mahlos in V). So $x \in V[G \restriction \lambda]$. Hence $x \in V_{j(\kappa)}[G \restriction \lambda] \subset M[G_1][\widetilde{H}_1]$. □

This theorem shows that we cannot prove there is a successor cardinal κ^+ such that $\omega_1 \to \kappa^+$ from a superstrong cardinal. Does a supercompact cardinal imply the existence of such a κ^+?

Acknowledgment

The author thanks Ernest Schimmerling and James Cummings for helpful discussions.

References

[B] J. Baumgartner, *Iterated forcing* (A. R. D. Mathis, ed.), Surveys in Set Theory, Cambridge Univ. Press, Cambridge, 1983.

[F] M. Foreman, *Potent axioms*, Trans. Amer. Math. Soc. **294** (1986), 1–28.

[J] T. J. Jech, *Set theory*, Academic Press, New York, 1978.

[KM] A. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, Lecture Notes in Math., vol. 669, Springer-Verlag, Berlin and New York, 1978, pp. 99–275.

[LMS] J.-P. Levinski, M. Magidor, and S. Shelah, *Changs conjecture for* $\aleph_{\omega+1}$, Israel J. Math. **69** (1990), 161–172.

[MS] D. A. Martin and J. R. Steel, *A proof of projective determinancy*, J. Amer. Math. Soc. **2** (1989), 71–125.

[MSS] W. J. Mitchell, E. Schimmerling, and J. R. Steel, *The covering lemma up to one Woodin cardinal* (to appear).

[MiS] W. J. Mitchell and J. R. Steel, *Fine structure and iteration trees*, Lecture Notes in Logic, vol. 3, Springer-Verlag, Berlin, 1994.

[S] E. Schimmerling, *Combinatorial principles in the core model for one Woodin cardinal*, Ph.D. thesis, UCLA, 1992.

[T] J. Tryba, *No Jónsson filters over* $\aleph_\omega$, J. Symbolic Logic **52** (1987), 51–53.

[W88] H. Woodin, *Supercompact cardinals, sets of reals and weakly homogeneous trees*, Proc. Nat. Acad. Sci. U.S.A. **85** (1988), 6587–6591.

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