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ON ABSTRACT FUBINI THEOREMS FOR FINITELY ADDITIVE INTEGRATION

E. DE AMO AND M. DÍAZ CARRILLO

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ABSTRACT. A Fubini theorem for positive linear functionals on the vector lattice of the real-valued functions is given. This result properly contains that of the Riemann-μ-abstract integral.

INTRODUCTION

In [3] one starts with a functional $I: B \to \mathbb{R}$, defined on the vector lattice B of real-valued functions on a set X and assumed to be positive linear. One defines the extended function class $R_1(B, I)$ and extends I to $R_1(B, I)$ via one or the other of three classical methods (certain limits of elementary functions, equality of the upper and lower integrals, closure of B with respect to an R_1 type seminorm), and one gets the convergence theorems using a suitable "local convergence in measure". Riemann-μ, abstract Riemann-Loomis and Bourbaki integrals are subsumed.

In [5] Elsner has given a Fubini type theorem for the abstract Riemann-μ-integral. In this note, the integral extension of Lebesgue power introduced in [2] and [3] is used to develop a Fubini type theorem in quite general settings.

Let I_1 and I_2 be positive linear functionals on vector lattices over X_1 and X_2, respectively. A methodological simplification is obtained by constructing the iterated integrals, via a suitable extension of the linear functionals. Conditions are determined for an integrable function $f: X_1 \times X_2 \to \overline{\mathbb{R}}$, without assuming continuity, so that the iterated integrals exist and are equal. So, our results are a reasonable substitute for Fubini's theorem for finitely additive integration (or for corresponding to the analogues to the Daniell extension process, but without continuity assumptions on the elementary integral I).

1. PRODUCT SYSTEMS

On the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ we adopt the usual conventions $0(\pm\infty) := 0$ and $\infty + (-\infty) := 0$. We denote $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$, $a, b \in \overline{\mathbb{R}}$.

Terminology and notation used are similar to that of [2], [3] and [10].

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(1) Throughout this note we shall assume that for $j = 1, 2$, X_j is an arbitrary set, $B_j \subset R^{X_j}$ a vector lattice (with respect to pointwise operations) and $I_j: B_j \to \mathbb{R}$ a linear functional which is positive, i.e., $I_j(f) \geq 0$ for all $f \geq 0$ in B_j.

Let $X_3 := X_1 \times X_2$ and $B_3 \subset R^{X_3}$ a vector lattice. For $f \in \overline{\mathbb{R}}^{X_3}$ and for $x \in X_1$ we define the function f_x on X_2 by $f_x(y) = f(x, y)$ for each $y \in X_2$. Let f be a function on X_3 such that $f_x \in B_2$ for each $x \in X_1$. Then setting $(I_2 f)(x) := I_2(f_x)$ for each $x \in X_1$, we have defined the function $I_2 f$ on X_1.

(2) A system (X_3, B_3) is called a *product system* with respect to (X_1, B_1) and (X_2, B_2), whenever for each $f \in B_3$ the following conditions are satisfied:

(i) $f_x \in B_2$ for each $x \in X_1$.

(ii) $I_2 f \in B_1$.

In all that follows (X_3, B_3) will be a product system. We define a positive linear functional on B_3 by the rule $I_3(f) := I_1(I_2 f)$ for each $f \in B_3$.

2. The abstract Fubini theorem

2.1. **Proper Riemann integration.** (3) For $f \in \overline{\mathbb{R}}^{X_j}$, $j = 1, 2, 3$, we define *Riemann upper* and *lower integrals* by

$I_j^-(f) := \inf\{I_j(h); f \leq h \in B_j\}$, with $\inf \varnothing := \infty$ and $I_j^+(f) := -I_j^-(-f)$.

I_j^- is positively homogeneous and subadditive on $\overline{\mathbb{R}}^{X_j}$.

For $f \in \overline{\mathbb{R}}^{X_3}$ we define the function $I_2^- f: X_1 \to \overline{\mathbb{R}}$ by $(I_2^- f)(x) := I_2^-(f_x)$ for each $x \in X_1$. Similarly, $(I_2^+ f)(x) := -I_2^-(-f_x)$.

Lemma 1. *If $f \in \overline{\mathbb{R}}^{X_3}$, then $I_1^-(I_2^- f) \leq I_3^-(f)$ and $I_3^+(f) \leq I_1^+(I_2^+ f)$.*

Proof. By (3) and (ii) of (2), one has $I_3^-(f) := \inf\{I_1(I_2 h);\ f \leq h \in B_3\} \geq \inf\{I_1(I_2 h);\ I_2^- f \leq I_2^- h = I_2 h, h \in B_3\} \geq \inf\{I_1(g); I_2^- f \leq g \in B_1\} =: I_1^-(I_2^- f)$.

The rest of the proof is similar. □

(4) The set $R_{\text{prop}}(B_j, I_j)$ of *proper Riemann integrable functions* is defined as the set of those functions $f \in \mathbb{R}^{X_j}$ such that any one of the following conditions, which are equivalent, is satisfied.

(i) Given any $\varepsilon \in \mathbb{R}^+$, there exist $h, g \in B_j$ such that $I_j(h - g) < \varepsilon$, with $g \leq f \leq h$.

(ii) $I_j^+(f) = I_j^-(f) \in \mathbb{R}$.

We have that $R_{\text{prop}}(B_j, I_j)$ is the closure of B_j with respect to the integral seminorm $I_j^-(|\cdot|)$. If $f \in R_{\text{prop}}(B_j, I_j)$, $I_j(f) := I_j^+(f) = I_j^-(f)$ (see, for example, [1], [2]).

$A \subset X_j$ is called an *I_j^--null set* iff $I_j^-(\chi_A) = 0$.

Theorem 1. *If $f \in R_{\text{prop}}(B_3, I_3)$, then:*

(i) *$I_2^- f$, $I_2^+ f \in R_{\text{prop}}(B_1, I_1)$.*

(ii) *There exist $A_k \subset X_1$, $k \in \mathbb{N}$, I_1^--null sets, such that $f_x \in R_{\text{prop}}(B_2, I_2)$ for all $x \in X_1 - \bigcup_1^\infty A_k$.*

(iii) *There exists $g \in R_{\text{prop}}(B_1, I_1)$ defined by $I_2^-(f_x)$ if $f_x \in R_{\text{prop}}(B_2, I_2)$, and such that $I_3(f) = I_1(g)$.*

Proof. (i) For $f \in R_{\text{prop}}(B_3, I_3)$, by (4), (3) and Lemma 1, we have

$$I_3(f) := I_3^-(f) \geq I_1^-(I_2^- f) \geq \left\{ \begin{matrix} I_1^-(I_2^+ f) \\ I_1^+(I_2^- f) \end{matrix} \right\} \geq I_1^+(I_2^+ f) \geq I_3^+(f) := I_3(f).$$

Then, $I_3(f) = I_1^-(I_2^- f) = I_1^+(I_2^- f) \in \mathbb{R}$, and by (4) $I_2^- f \in R_{\text{prop}}(B_1, I_1)$. Similarly, $I_2^+ f \in R_{\text{prop}}(B_1, I_1)$.

(ii) For $x \in X_1$, set $h(x) := I_2^-(f_x) - I_2^+(f_x)$. One has $0 \leq h \in R_{\text{prop}}(B_1, I_1)$ and $I_1(h) = 0$.

Now, let $A_k := \{x \in X_1 ; h(x) \geq \frac{1}{k}\}$, $k \in \mathbb{N}$. Since $I_1^-(\chi_{A_k}) \leq kI_1(h) = 0$, A_k are I_1^--null sets, and by (4), $f_x \in R_{\text{prop}}(B_2, I_2)$ for all $x \in X_1 - \bigcup_1^\infty A_k$.

(iii) Finally, if $g \in \overline{\mathbb{R}}^{X_1}$ such that $I_2^-(f_x) \leq g(x) \leq I_2^+(f_x)$ for all $x \in X_1 - \bigcup_1^\infty A_k$, then, by (4), we obtain $g \in R_{\text{prop}}(B_1, I_1)$ and $I_1(g) = I_1^-(I_2^- f) = I_3(f)$. □

Observe that if $l \in R_{\text{prop}}(B_1, I_1)$ such that $l(x) = I_2^-(f_x)$ whenever $f_x \in R_{\text{prop}}(B_2, I_2)$, then $I_1(l) = I_3(f)$ and $I_1^-(|g - l|) = 0$.

Remarks 1. 1. In general $I_3^+(f) \geq I_1^+(I_2^+ f)$ is false for all $f \in \overline{\mathbb{R}}^{X_3}$, by 3.4 and Example 2 below. Therefore, an analogue to Theorem 1 for summable functions of [2] is in general not true.

Nevertheless, if I is monotone-net-continuous = Bourbaki's continuity condition, then Daniell $L^1(B, I) \subset$ Bourbaki extension L^τ and $I^+ =$ upper Bourbaki extension I^τ. In this case, $I_3^\tau \geq I_1^\tau(I_2^\tau)$ and there is an analogue to Theorem 1 (see [2], [6] and [13], p. 186).

2. For arbitrary I/B it is easy to show that if $(h_n) \subset B$, $0 \leq h_{n+1} \leq h_n$, $n \in \mathbb{N}$, $I(h_n) \to 0$, as $n \to \infty$, then there exist $A_k \subset X$ such that $I^-(\chi_{A_k}) = 0$, and that if $x \notin \bigcup_1^\infty A_k$, then $h_n(x) \to 0$, as $n \to \infty$.

In general, $\bigcup_1^\infty A_k$ is not an I^--null set by 3.4 and Example 3 below. If I is σ-continuous, then $\bigcup_1^\infty A_k$ is an I^σ-null set (see [9], p. 265).

2.2. **Abstract Riemann integration.** (5) For any $f \in \overline{\mathbb{R}}^{X_j}$, $j = 1, 2, 3$, the corresponding localized functionals in the sense of Schäfke [14] are defined by

$$I_{j,l}^-(f) := \sup\{I_j^-(f \wedge h) ; 0 \leq h \in B_j\},$$

and for $f \in \overline{\mathbb{R}}^{X_3}$, $I_{2,l}^- f \colon X_1 \to \overline{\mathbb{R}}$ is defined by $(I_{2,l}^- f)(x) := I_{2,l}^-(f_x)$ for each $x \in X_1$. $I_{j,l}^-$ is monotone and subadditive on $\overline{\mathbb{R}}^{X_j}$.

In view of the definitions involved, we have

(6) $(I_{j,l}^-)_{j,l} = I_{j,l}^- \leq I_j$, and $I_{j,l}^-(f) = I_j(f)$ if $f \in \overline{\mathbb{R}}^{X_j}$ and $f \leq$ some $h \in B_j$.

(7) For $j = 1, 2, 3$, the set $R_1(B_j, I_j)$ of I_j*-integrable functions* is defined as the closure of B_j in $\overline{\mathbb{R}}^{X_j}$ with respect to the integral seminorm $I_{j,l}^-(|\ |)$.

$R_1(B_j, I_j)$ is closed with respect to $\pm, \alpha \cdot (\alpha \in \mathbb{R})$; $|\ |$, $\wedge$, $\vee$. $I_{j,l}^-/R_1(B_j, I_j)$ is a positive linear functional (= unique $I_{j,l}^-$-continuous extension of I_j/B_j).

By [3], $R_1(B_j, I_j)$ is the set of all $f \in \overline{\mathbb{R}}^{X_j}$ to which there exists a sequence $(h_n) \subset B_j$, which is a Cauchy sequence with respect to $I_j(|\cdot|)$ and with $h_n \to f(I_j^-)$; then $I_j(f) := \lim I(h_n)$, $n \to \infty$. In general, $R_{\text{prop}} \subset R_1$ with coinciding integrals, and $\subset$ generally strict. For further properties of R_1 see [3] and [10].

In all that follows we assume the following condition

(*)

To $h \in B_1$, $g \in B_2$ there exists $l \in B_3$ such that $g(y) \leq l(x, y)$ if $h(x) > 0$.

Lemma 2. *Let* $f \in \overline{\mathbb{R}}^{X_3}$ *such that the following condition holds:*

$$(**) \qquad |f_x| \leq g \in B_2 \quad \text{for each } x \in X_1 .$$

Then, $I_{1,l}^-(I_{2,l}^- f) \leq I_{3,l}^-(f)$.

Proof. For $f_x \in \overline{\mathbb{R}}_+^{X_2}$, $x \in X_1$, we have with $(**)$ and (5) $(I_{2,l}^- f)(x) = (I_2^- f)(x)$, so that $I_{1,l}^-(I_{2,l}^- f) \leq I_{1,l}^-(I_2^- f) := \sup\{I_1^-((I_2^- f) \wedge h) ; 0 \leq h \in B_1\}$.

Now, with $(*)$, $(I_2^- f) \wedge h \leq I_2^- f \leq I_2^- f \wedge l$, where $l \in B_3$ and $f_x \leq g \leq l_x$ for each $x \in X_1$, $h(x) > 0$. Hence, with Lemma 1 and (5), we have $I_1^-((I_2^- f) \wedge h) \leq I_1^-(I_2^- f \wedge l) \leq I_3^-(f \wedge l) \leq I_{3,1}^-(f)$, for all $0 \leq h \in B_1$, and we conclude the result. □

Without $(**)$ Lemma 2 becomes false by p. 270 of [5]. Here there exists $f \in \overline{\mathbb{R}}_+^{X_3}$ such that $I_{1,l}^-(I_{2,l}^- f) = \infty$ and $I_{3,l}^-(f) = 0$.

Theorem 2 is obtained now in a similar way as Satz p. 141 of Hoffman [11] (see also Elsner [5]).

Theorem 2. *Let* (X_3, B_3) *be a product system, and let* $f \in R_1(B_3, I_3)$ *satisfying* $(**)$. *Then the following assertions hold:*

(i) *There exist* $A_k \subset X_1$, $k \in \mathbb{N}$, $I_{1,l}^-$*-null sets, such that* $f_x \in R_1(B_2, I_2)$ *for each* $x \in X_1 - \bigcup_1^\infty A_k$.

(ii) *There exists* $g \in R_1(B_1, I_1)$ *defined by* $I_{2,l}^-(f_x)$ *if* $f_x \in R_1(B_2, I_2)$, *and such that* $I_{1,l}^-(g) = I_{3,l}^-(f)$, *i.e.* $I_{3,l}^-(f) = I_{1,l}^-(I_{2,l}^- f)$.

Proof. (i) By (7), for $f \in R_1(B_3, I_3)$, given $\varepsilon > 0$ there exists $g \in B_3$ such that $I_{3,l}^-(|f - g|) < \varepsilon$.

For each $x \in X_1$ set $\varphi(x) := \inf\{I_{2,l}^-(|f_x - h|)$, for all $h \in B_2\}$ and set $A_k := \{x \in X_1 ; \varphi(x) \geq \frac{1}{k}\}$, $k \in \mathbb{N}$. By virtue of Lemma 2, it can be easily proved that the sets A_k, $k \in \mathbb{N}$, are $I_{1,l}^-$-null, and (i) follows immediately.

To prove (ii) it suffices to see that there is $(I_{2,l}^- g_n) \subset B_1$ such that $I_{2,l}^- g_n \to g(I_{1,l}^-)$, where $(g_n) \subset B_3$ and $I_{3,l}^-(|g_n - f|) \to 0$, as $n \to \infty$.

In fact, a calculation analogous to the proof given in [11] (Hauptsatz, p. 139, with "Fubini-integral norms"), and having in mind the properties stated in (6) and (7), permits to show the inequality

$$I_{1,l}^-(|(I_{2,l}^- g_n) - g|) \leq 3 I_{1,l}^-(I_{2,l}^- |g_n - f|) \leq 3 I_{3,l}^-(|g_n - f|).$$

Besides, $I_{1,l}^-(I_{2,l}^- g_n) \to I_{1,l}^-(g)$, as $n \to \infty$, and $I_{1,l}^-(g) = I_{3,l}^-(f)$. □

Remarks 2. 1. In the above statement usually all assumptions are essential. There exist counterexamples for the $\lambda \times \mu$-case (additive measure space, see 3.1. below) in [5], Bem.4.b, and 4.c.p.270. Similar examples show that one cannot substitute $|f_x| \leq g$ for $f_x \leq g$ in $(**)$.

2. There are simple examples of $f \in R_1(\lambda \times \mu, \mathbb{R})$ with $(**)$, but $f \notin R_{\text{prop}}(\lambda \times \mu, \mathbb{R})$:

$$f = \chi_{X_1 \times M}, \ M \in \text{ ring } \Omega_2, \ \text{ with } \mu(X_1) = \infty.$$

3. Let us finally remark that our results can be reformulated for Banach space-valued functions, using $f \cap g := \|f\|^{-1}(\|f\| \wedge g)f$, with $f: X \to E =$ Banach space, $g \in \mathbb{R}_+^X$, of [9], p. 327.

3. Applications and examples

1. If Ω is a semiring of sets $\subset X$ and $\mu: \Omega \to [0, \infty[$ is additive, then $B = B_\Omega :=$ real-valued step functions over Ω and $I = I_\mu := \int \cdot d_\mu$ satisfying (1).

Then the *proper Riemann-μ-integrable functions* $R_{\text{prop}}(\mu, \mathbb{R}) = I_\mu^-$-closure of B_Ω in $\mathbb{R}^X$, in the sense of Aumann [1], p. 448.

The space of *abstract Riemann-μ-integrable functions* $R_1(\mu, \mathbb{R})$ was presented essentially by Loomis [12]. For Banach space-valued functions it has been introduced by Dunford-Schwartz [4], and in more general form by Günzler [8], [9]. $R_{\text{prop}}(\mu, \mathbb{R}) \subset$ Dunford-Schwartz integral $L(X, \Omega, \mu, \mathbb{R}) \subset R_1(\mu, \overline{\mathbb{R}})$, with coinciding integrals; all $\subset$ are in general strict (see Lemma 9 of [10] and [9], pp. 199, 70).

In Gould [7], Stone's axiom $B \wedge 1 \subset B$ is assumed, so by [8] his results are already subsumed by the abstract Riemann integral (see, for example, [9], pp. 57, 268).

If Ω_1 and Ω_2 are semirings of sets from X_1 and X_2, and μ_1 and μ_2 are additive measures on Ω_1 and Ω_2, respectively, one can construct a product additive measure μ_3 in the set $X_3 := X_1 \times X_2$ and the induced integral I_{μ_3}.

If we set $\Omega_3 := \{A_1 \times A_2; A_j \in \Omega_j, j = 1, 2\}$, then, $\mu_3(A_1 \times A_2) := \mu_1(A_1) \cdot \mu_2(A_2) = I_{\mu_3}(\chi_{A_1 \times A_2})$. See [13], §16; [11], p. 125.

2. If $B = B_\Omega$ with $\Omega = \sigma$-ring and $I = I_\mu$ with μ σ-additive, then $R_1(\mu, \overline{\mathbb{R}}) = L^1(\mu, \overline{\mathbb{R}})$ (:= Lebesgue-μ-integrable functions), and $f_n \to f$ μ-almost everywhere implies $f_n \to f(I_\mu^-)$ for μ-measurable f_n, by [9], p. 265; and we get the usual Lebesgue convergence theorems.

In [5] Elsner has given a very thorough and interesting treatment of the Fubini theorem for the abstract Riemann-μ-integral. Our results contain properly that of [5], for which we obtain simplified proofs. Indeed, Example 1 below shows that there exist functions for which Theorem 1 is applicable, but not the corresponding result of [5] or even [11], p. 129.

Observe that for the $\lambda \times \mu$-case, $(*)$ holds and $(**)$ means that $|f|$ is bounded and there exists $P \in$ ring generated Ω_2 such that $\text{supp}(f) \subset X_1 \times P$.

In [14] integration with local Loomis-Schäfke integral seminorms is obtained. With (7), we have $R_1(B, I) \cap \mathbb{R}^X =$ Schäfke local I_1^--closure of B, and $R_1(B, I) \cap \mathbb{R}^x =$ "one-sided completion" of Loomis [12], p. 170.

3. We denote by $B_1 \otimes B_2$ the vector space of functions on X_3 generated by the family $\{g \otimes h; g \in B_1, h \in B_2\}$, where $(f \otimes k)(x, y) := f(x) \cdot k(y)$ for arbitrary f and k on X_1 and X_2, respectively. If $|f| \in B_1 \otimes B_2$ whenever $f \in B_1 \otimes B_2$, then $B_1 \otimes B_2$ is a product system with respect to (B_1, I_1) and (B_2, I_2) (see [13], §15; [6], p. 187), and Sections 1 and 2 are applicable.

4. Using Examples 1–3 below it is not difficult to check that there are finitely additive λ, μ on rings and f, g, h_n with $0 \le f = I_{\mu \times \mu}^-$-null functions, but

$f_x \in R_{\text{prop}}(\mu, \mathbb{R})$ for no $x \in X_1$ (Example 1); $g \in R_1(\lambda \times \mu, \mathbb{R})$, all $g^y \in B_1$, $I_1(g^y) \in B_2$, but $\int g d(\lambda \times \mu) \neq \int(\int g d\lambda) d\mu$ (Example 2); $0 \leq h_{n+1} \leq h_n$, $\int h_n d\mu \to 0$, but $h_n(x) \to 0$ for no x (Example 3). g can be found in Elsner [5], p. 270.

Example 1. Let $X_1 = X_2 = \mathbb{N}$, $\Omega_1 = \Omega_2 = \{\mathbb{N} - E, E;$ finite set $\subset \mathbb{N}\}$, $\mu_1 = \mu_2 = \mu$ finitely additive measure, such that $\mu(E) := 0$, $\mu(\mathbb{N}) := 1$.

Let $X_3 := \mathbb{N} \times \mathbb{N}$, $\Omega := \{X_3 - E, E; E$ finite set $\subset X_3\}$, and $\nu\colon \Omega \to \mathbb{R}$, $\nu(E) := 0$, $\nu(X_3) := 1$.

Example 2. Let $X_1 = \mathbb{R}$, $\Omega_1 := \{]a, b]; a, b \in \mathbb{R}, a \leq b\}$, $\lambda(]a, b]) := b - a$; $X_2 = \mathbb{N}$, $\Omega_2 := \{\mathbb{N} - E, E; E$ finite set $\subset \mathbb{N}\}$, $\mu(\mathbb{N}) = 1$, $\mu(E) = 0$, and $X_3 := X_1 \times X_2 = \mathbb{R} \times \mathbb{N}$. Let $I_1 = I_\lambda$, $I_2 = I_\mu$, $B_1 = B_{\Omega_1}$, $B_2 = B_{\Omega_2}$, $B_3 := B_{\Omega_1 \times \Omega_2}$, $I_3 := I_1 \circ I_2 = \int \cdot d(\lambda \times \mu)$.

Example 3. Let $X = \mathbb{N}$, $\Omega = \{\mathbb{N} - E, E; E$ finite set $\subset \mathbb{N}\}$, $\mu(\mathbb{N}) = 1$, $\mu(E) = 0$, $B = B_\Omega$, and $I = I_\mu$.

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Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain