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TWO-DIMENSIONAL REPRESENTATIONS OF UNIFORM ALGEBRAS

TAKAHIKO NAKAZI AND KATSUTOSHI TAKAHASHI

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ABSTRACT. It is shown that every two-dimensional representation of a uniform algebra has a dilation, which extends the result by Paulsen. We also prove some dilation result for a representation of the disk algebra.

1. INTRODUCTION

Let $C(X)$ be the algebra of complex-valued continuous functions on a compact Hausdorff space X, and let A be a uniform algebra on X. Let $L(H)$ denote the algebra of all bounded linear operators on a separable Hilbert space H. An algebra homomorphism $\Phi: A \to L(H)$ is called a representation of A on H if $\Phi(1) = I_H$ and Φ is contractive, i.e., $\|\Phi(f)\| \le \|f\|$ for all $f \in A$. Two representations $\Phi_1: A \to L(H_1)$ and $\Phi_2: A \to L(H_2)$ are said to be unitarily equivalent if there exists a unitary operator $U: H_1 \to H_2$ such that $U\Phi_1(f) = \Phi_2(f)U$ for all $f \in A$. For a representation Φ of A on H, a representation $\tilde{\Phi}: C(X) \to L(K)$ is called a dilation of Φ if $H \subset K$ and $\Phi(f) = P_H\tilde{\Phi}(f)|H$ for all $f \in A$, where P_H is the orthogonal projection of K onto H. Paulsen [6] showed that every two-dimensional representation of A has a dilation in the case where A is the algebra of all functions uniformly approximated on a compact subset X of the complex plane by rational functions with poles off X (see also [5]). In this note we give another proof of the above dilation result (for a general uniform algebra A).

B. Cole (see [1]) showed that for any closed ideal J in a uniform algebra A, the quotient algebra A/J is isometrically isomorphic to an algebra of bounded operators on a Hilbert space H, or equivalently, there is a representation $\Phi: A \to L(H)$ such that $\|\Phi(f)\| = \|f + J\|$ for all $f \in A$, where $\|f + J\|$ is the quotient norm of the coset $f + J$ of f in A/J. We say a representation Φ of A is Q-isometric if $\|\Phi(f)\| = \|f + \ker\Phi\|$ for all $f \in A$, and a Q-isometric representation $\tilde{\Phi}: A \to L(K)$ is a Q-isometric dilation of a representation $\Phi: A \to L(H)$ if $H \subset K$ and $\Phi(f) = P_H\tilde{\Phi}(f)|H$ for all $f \in A$. A

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Q-isometric representation of A is used by Cole, Lewis and Wermer [2] to generalize Pick's conditions of the interpolation problem for the disk algebra to the case of the uniform algebra A. The result of Cole stated above shows that any representation Φ of A has a Q-isometric dilation. Indeed, by Cole's result, there exists a Q-isometric representation Ψ such that $\ker\Psi = \ker\Phi$. Then the representation $\tilde{\Phi}$ defined by $\tilde{\Phi}(f) = \Phi(f) \oplus \Psi(f)$ $(f \in A)$ is a Q-isometric dilation of Φ. It also follows from our proof of the dilation result (Theorem 1) that if a representation $\Phi\colon A \to L(H)$ satisfies $\dim(A/\ker\Phi) = 2$, then Φ has a Q-isometric dilation $\tilde{\Phi}\colon A \to L(K)$ which is minimal in the sense that $K = \bigvee_{f\in A} \tilde{\Phi}(f)H$. In Section 3 it is shown that every representation of the disk algebra has a minimal Q-isometric dilation.

2. Two-dimensional representations

In this section we prove the following theorem, which extends the result by Paulsen [6].

Theorem 1. *Let $\Phi\colon A \to L(H)$ be a representation of A. If $\dim(A/\ker\Phi) = 2$, then Φ has a dilation.*

Using Misra's method [5], we first determine representations $\Phi\colon A \to L(H)$ such that $\dim(A/\ker\Phi) = 2$.

Let J be an ideal of A with $\dim(A/J) = 2$. Then

$$J = \{f \in A\colon f(x) = f(y) = 0\}, \tag{1}$$

where x and y are two points in the maximal ideal space $M(A)$ of A, or

$$J = \{f \in A\colon f(x) = \delta(f) = 0\}, \tag{2}$$

where $x \in M(A)$ and δ is a bounded point derivation at x, that is, δ is a bounded linear functional on A such that $\delta(fg) = f(x)\delta(g) + g(x)\delta(f)$ for $f, g \in A$ (see, e.g., [3]).

Lemma 1. *Let $\Phi\colon A \to L(H)$ be a homomorphism with $\Phi(1) = I_H$ and assume that $\dim(A/\ker\Phi) = 2$. Then, according as $J = \ker\Phi$ is of the form (1) or (2), $\Phi(f)$ is expressed as*

$$\Phi(f) = \begin{pmatrix} f(x)I_{H_1} & (f(x)-f(y))C \\ 0 & f(y)I_{H_2} \end{pmatrix} \quad \text{on } H = H_1 \oplus H_2 \tag{3}$$

or

$$\Phi(f) = \begin{pmatrix} f(x)I_{H_1} & \delta(f)C \\ 0 & f(x)I_{H_2} \end{pmatrix} \quad \text{on } H = H_1 \oplus H_2 \tag{3'}$$

for all $f \in A$, where C is a bounded linear operator from H_2 to H_1.

Proof. Suppose that J is of the form (1). Take functions f_1 and f_2 in A such that $f_1(x) = f_2(y) = 1$ and $f_1(y) = f_2(x) = 0$. Then $\Phi(f_1)$ is idempotent and so

$$\Phi(f_1) = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix} \text{ on } H = \operatorname{ran}\Phi(f_1) \oplus (\operatorname{ran}\Phi(f_1))^{\perp}.$$

Since $\Phi(f_1) + \Phi(f_2) = I$ and $f - f(x)f_1 - f(y)f_2 \in J$ for $f \in A$, we have

$$\Phi(f) = \begin{pmatrix} f(x)I & (f(x)-f(y))C \\ 0 & f(y)I \end{pmatrix} \quad \text{on } H = \operatorname{ran}\Phi(f_1) \oplus (\operatorname{ran}\Phi(f_1))^{\perp},$$

for all $f \in A$. For the case where J is of the form (2), take $f_0 \in A$ such that $f_0(x) = 0$ and $\delta(f_0) = 1$, and note that $\Phi(f_0)^2 = 0$ and $f - f(x) - \delta(f)f_0 \in J$ for $f \in A$.

Lemma 2 (cf. [5, the proof of Theorem 2.3]). *Let $C: H_2 \to H_1$ and $D: K_2 \to K_1$ be two operators, where H_1, H_2, K_1 and K_2 are Hilbert spaces. If $\|C\| \leq \|D\|$, then*

$$\left\| \begin{pmatrix} aI_{H_1} & C \\ 0 & bI_{H_2} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} aI_{K_1} & D \\ 0 & bI_{K_2} \end{pmatrix} \right\|$$

for any scalars a and b.

Proof. If $a = 0$ or $D = 0$, the inequality is clear. So suppose that a and D are nonzero. By considering $(1+\varepsilon)D$ $(\varepsilon > 0)$ instead of D, we can also assume that $\|C\| < \|D\|$. Take any unit vector $\binom{x}{y}$ in $H = H_1 \oplus H_2$ $(y \neq 0)$. Since $\|C\| < \|D\|$, there is $y' \in K_2$ such that $\|Cy\| < \|Dy'\|$ and $\|y'\| = \|y\|$. Set $x' = \frac{|a|}{a} \frac{\|x\|}{\|Dy'\|} Dy'$. Then $\|\binom{x'}{y'}\| = 1$, and we have

$$\left\| \begin{pmatrix} aI_{H_1} & C \\ 0 & bI_{H_2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| < \left\| \begin{pmatrix} aI_{K_1} & D \\ 0 & bI_{K_2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \right\|,$$

which implies the required inequality.

Let μ be a probability measure on X, and let $H^2(\mu)$ and $[J]_\mu$ denote the closure in $L^2(\mu)$ of A and of an ideal J, respectively. For each $f \in A$, we define an operator S_f^μ on $H = H^2(\mu) \ominus [J]_\mu$ by $S_f^\mu h = P_H(fh)$ for each $h \in H$. Then the map $\Phi^\mu: f \mapsto S_f^\mu$ is a representation of A on H such that $\ker \Phi^\mu \supset J$ and has a dilation $\tilde{\Phi}^\mu: f \mapsto M_f^\mu$, where for $f \in C(X)$, M_f^μ denotes the multiplication operator by f on $L^2(\mu)$. B. Cole (see [1]) showed that for each $f \in A$, there exists a probability measure ν such that $\|S_f^\nu\| = \|f + J\|$.

For $x, y \in M(A)$ and a bounded point derivation δ at x, let

$$\sigma(x, y) = \sup\{|f(y)|: f(x) = 0 \text{ and } \|f\| \leq 1\}$$

and

$$\rho(x, \delta) = \sup\{|\delta(f)|: f(x) = 0 \text{ and } \|f\| \leq 1\}.$$

Lemma 3 (cf. [5, Theorem 1.1 and Corollary 1.1]). *Let $\Phi: A \to L(H)$ be a homomorphism with $\Phi(1) = I$ such that $\dim(A/\ker\Phi) = 2$, and let C be as in Lemma 1. Then Φ is a representation of A on H if and only if, according as $J = \ker\Phi$ is of the form (1) or (2),*

$$\|C\| \leq \left(\frac{1}{\sigma(x, y)^2} - 1\right)^{1/2} \text{ or } \rho(x, \delta)^{-1}. \tag{4}$$

Furthermore, the equality in (4) holds if and only if $\|\Phi(f)\| = \|f + J\|$ for all $f \in A$.

Proof. By [5, Remark 2], the condition that Φ is contractive is equivalent to the condition that $\|\Phi(f)\| \leq \|f\|$ for all $f \in J_x = \{f: f(x) = 0\}$. Since $\dim(J_x/J) = 1$ by assumption, the latter is equivalent to the condition that $\|\Phi(f)\| \leq \|f + J\|$ for some $f \in J_x \backslash J$. In the case where J is of the form (1), for $f \in J_x \backslash J$, by (3) we have

$$\Phi(f)^*\Phi(f) = \begin{pmatrix} 0 & 0 \\ 0 & |f(y)|^2(C^*C + I) \end{pmatrix},$$

hence

$$\|\Phi(f)\|^2 = |f(y)|^2(\|C\|^2 + 1) = \sigma(x, y)^2(\|C\|^2 + 1)\|f + J\|^2.$$

Similarly, for the case where J is of the form (2), we have

$$\|\Phi(f)\| = \rho(x, \delta)\|C\| \, \|f + J\|$$

for $f \in J_x \backslash J$. Hence the first part follows. Also, if $\|\Phi(f)\| = \|f + J\|$ for $f \in J_x \backslash J$, then it follows that Φ is contractive and the equality in (4) holds. Conversely, assume that the equality in (4) holds. By Cole's result, for each $f \in A$, there is a probability measure ν such that $\|f + J\| = \|S_f^\nu\|$. Since the map $\Phi^\nu : g \mapsto S_g^\nu$ is a representation of A such that $\ker \Phi^\nu \supset J$, it follows from the first part and Lemma 2 that $\|S_f^\nu\| \le \|\Phi(f)\|$. (Note that if $\dim(A/\ker \Phi^\nu) = 1$, then S_f^ν is the operator of multiplication by $f(x)$ or $f(y)$ on the one-dimensional space and so $\|S_f^\nu\| \le \|\Phi(f)\|$.) Therefore $\|f + J\| = \|\Phi(f)\|$ for all $f \in A$.

Corollary 1. *Let J be an ideal of A such that* $\dim(A/J) = 2$. *Then there is a probability measure μ such that $\|S_f^\mu\| = \|f + J\|$ for all $f \in A$.*

Proof. The ideal J is of the form (1) or (2). Take an $f \in A \backslash J$ such that $f(x) = 0$. By Cole's result, there exists a probability measure μ such that $\|f + J\| = \|S_f^\mu\|$. The map $\Phi^\mu : g \mapsto S_g^\mu$ is a representation of A such that $\ker \Phi^\mu \supset J$. If $\ker \Phi^\mu = J$, then it follows from Lemma 3 (and its proof) that μ is the required measure. On the other hand, if $\ker \Phi^\mu \ne J$, then, since $S_f^\mu \ne 0$, the ideal J is of the form (1) and $S_f^\mu = f(y)$. It follows that $\|f + J\| = |f(y)|$ ($\ne 0$), hence $\sigma(x, y) = 1$, which means x and y belong to the different Gleason parts of $M(A)$. In this case, by Lemma 3 any representation Φ of A such that $\ker \Phi = J$ satisfies $\|\Phi(g)\| = \|g + J\|$ for all $g \in A$. Therefore we have only to take a probability measure μ such that $\dim(H^2(\mu) \ominus [J]_\mu) = 2$, for example, $\mu = (\nu_1 + \nu_2)/2$, where ν_1 and ν_2 are representing measures of x and y, respectively.

Proof of Theorem 1. Suppose that $J = \ker \Phi$ is of the form (1). By Lemma 3, $\Phi(f)$ $(f \in A)$ is expressed as (3) with $\|C\| \le \alpha = (\sigma(x, y)^{-2} - 1)^{1/2}$. If $\alpha = 0$, then $C = 0$ and clearly Φ has a dilation, which is unitarily equivalent to the representation

$$f \mapsto \left(\sum_{1 \le n \le d_1} \oplus M_f^{\mu_1} \right) \oplus \left(\sum_{1 \le n \le d_2} \oplus M_f^{\mu_2} \right)$$

of $C(X)$ on the space $(\sum_{1 \le n \le d_1} \oplus L^2(\mu_1)) \oplus (\sum_{1 \le n \le d_2} \oplus L^2(\mu_2))$, where μ_1 and μ_2 are representing measures of x and y, respectively, and $d_i = \dim H_i$ for $i = 1, 2$. So assume $\alpha \ne 0$. Then we can define an operator

$$W = \begin{pmatrix} (I_{H_1} - \alpha^{-2} C C^*)^{1/2} & 0 \\ \alpha^{-1} C^* & 0 \\ 0 & I_{H_2} \end{pmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2 \oplus H_2.$$

Also, define a representation Ψ of A on $K = H_1 \oplus H_2 \oplus H_2$ by

$$\Psi(f) = \begin{pmatrix} f(x) I_{H_1} & 0 & 0 \\ 0 & f(x) I_{H_2} & \alpha(f(x) - f(y)) I_{H_2} \\ 0 & 0 & f(y) I_{H_2} \end{pmatrix}.$$

Then the operator W is isometric and satisfies $\Psi(f)^* W = W\Phi(f)^*$ for $f \in A$. Therefore $\operatorname{ran} W$ is invariant for the algebra $\{\Psi(f)^* : f \in A\}$ and the representation Φ is unitarily equivalent to a representation Ψ_0 of A on $\operatorname{ran} W$ defined by $\Psi_0(f) = P_{\operatorname{ran} W}\Psi(f)|\operatorname{ran} W$. By Corollary 1 and Lemma 3, there exists a probability measure μ such that for $f \in A$, the operator S_f^μ on $H^2(\mu) \ominus [J]_\mu$ is expressed as

$$S_f^\mu = \begin{pmatrix} f(x) & \alpha(f(x) - f(y)) \\ 0 & f(y) \end{pmatrix}$$

(with respect to some orthonormal basis). Also, if ν is a representing measure of x, then S_f^ν is the multiplication operator by $f(x)$ on the one-dimensional space. Thus Ψ has a dilation, which is unitarily equivalent to the representation

$$f \mapsto \left(\sum_{1 \le n \le d_1} \oplus M_f^\nu \right) \oplus \left(\sum_{1 \le n \le d_2} \oplus M_f^\mu \right)$$

of $C(X)$ on $(\sum_{1 \le i \le d_1} \oplus L^2(\nu)) \oplus (\sum_{1 \le i \le d_2} \oplus L^2(\mu))$. Hence it follows that Φ has a dilation.

The above argument is also applied to the case where J is of the form (2), if the definition of $\Psi(f)$ is replaced by

$$\Psi(f) = \begin{pmatrix} f(x)I_{H_1} & 0 & 0 \\ 0 & f(x)I_{H_2} & \alpha\delta(f)I_{H_2} \\ 0 & 0 & f(x)I_{H_2} \end{pmatrix},$$

where $\alpha = \rho(x, \delta)^{-1}$ (> 0). Thus the proof is complete.

Corollary 2. *If* Φ *is a representation of* A *with* $\dim(A/\ker\Phi) = 2$, *then* Φ *has a minimal* Q*-isometric dilation.*

Proof. Let Ψ, Ψ_0 and W be as in the proof of Theorem 1. Then the invariant subspace $K_1 = \bigvee_{f \in A} \Psi(f) \operatorname{ran} W$ of the algebra $\{\Psi(f) : f \in A\}$ generated by $\operatorname{ran} W$ includes the space $\{0\} \oplus H_2 \oplus H_2$, hence the representation of $A : f \mapsto \Psi(f)|K_1$ is a minimal Q-isometric dilation of Ψ_0. Since Φ is unitarily equivalent to Ψ_0, it follows that Φ has a minimal Q-isometric dilation. (Note that if $\alpha = 0$, then Φ is Q-isometric by Lemma 3.)

3. Representations of the disk algebra

We consider a minimal Q-isometric dilation of a representation of the disk algebra. In the following, A denotes the disc algebra, i.e., A is the algebra of all continuous functions on the unit circle $\mathbf{T}$ whose Fourier coefficients vanish on the negative integers. Let H^p $(1 \le p \le \infty)$ denote the Hardy space on $\mathbf{T}$, thus H^p is the closure of A in $L^p = L^p(m)$ or the weak*-closure of A in $L^\infty = L^\infty(m)$ according as $p < \infty$ or $p = \infty$, where m is the Lebesgue measure of $\mathbf{T}$.

We use results from the dilation theory of Sz.-Nagy and Foias [8]. Let T be a contraction (i.e., $\|T\| \le 1$) on a Hilbert space H. Then, as is well known, T can be decomposed as $T = U \oplus T_1$ on $H = H_u \oplus H_1$ where U is a unitary operator on H_u and T_1 is a completely nonunitary contraction on H_1, that is, T_1 has no nonzero invariant subspace M such that $T_1|M$ is unitary (see [8, Chap. I, Theorem 3.2]). For a completely nonunitary contraction T on

H, the Sz.-Nagy and Foias functional calculus defines the weak*-continuous algebra homomorphism $\Phi_T: f \mapsto f(T)$ from H^∞ to $L(H)$, and T is said to be of class C_0 if Φ_T is not injective (see [8, Chap. III]). If T is of class C_0, then $T^{*n} \to 0$ strongly (see [8, Chap. III, Proposition 4.2]), thus T is unitarily equivalent to the (functional model) operator

$$S(M) = P_{H^2(E)\ominus M} S | H^2(E) \ominus M,$$

where $H^2(E)$ is the E-valued Hardy space (E is a Hilbert space), S is the unilateral shift on $H^2(E)$ and M is an invariant subspace of S such that $\bigvee_{n\geq 0} S^n(H^2(E)\ominus M) = H^2(E)$ (see [8, Chap. VI]). Also, since $\ker\Phi_T$ ($\neq \{0\}$) is a weak*-closed ideal in H^∞, we have $\ker\Phi_T = qH^\infty$ for an inner function q. The following lemma immediately follows from these facts.

Lemma 4. *If T is a contraction on H of class C_0, then there is a contraction $\widetilde{T}$ on $\widetilde{H}$ $(\supset H)$ of class C_0 satisfying the following conditions*:

(i) $T^* = \widetilde{T}^*|H$;
(ii) $\|f(\widetilde{T})\| = \|f + \ker\Phi_T\|$ *for all* $f \in H^\infty$;
(iii) $\widetilde{H} = \bigvee_{n\geq 0} \widetilde{T}^n H$.

Proof. We may consider T as the functional model $S(M) = P_{H^2(E)\ominus M} S|H^2(E) \ominus M$. Let $\ker\Phi_T = qH^\infty$, where q is inner. Since $q(S(M)) = 0$, we have $M \supset qH^2(E)$. Define a contraction $\widetilde{T}$ on $\widetilde{H} = H^2(E) \ominus qH^2(E)$ by $\widetilde{T}^* = S^*|\widetilde{H}$. Then clearly (i) holds and the condition $\bigvee_{n\geq 0} S^n(H^2(E)\ominus M) = H^2(E)$ implies (iii). Also, $\widetilde{T}$ is unitarily equivalent to the direct sum $\sum_{1\leq n\leq d} \oplus S(q)$, where $d = \dim E$ and $S(q)$ is an operator on $H^2 \ominus qH^2$ defined by $S(q)h = P_{H^2\ominus qH^2}(zh)$ $(h \in H^2 \ominus qH^2)$. Therefore, for $f \in H^\infty$, we have

$$\|f(\widetilde{T})\| = \left\| f\left(\sum_{1\leq n\leq d} \oplus S(q)\right)\right\| = \|f(S(q))\|,$$

and so $\|f(\widetilde{T})\| = \|f + qH^\infty\|$ (see [7]).

For a closed subset K of $\mathbf{T}$ (of measure zero), let $I(K)$ denote the ideal consisting of all functions of A which vanish on K. For each $f \in A$, $\|f + I(K)\| = \|f\|_K$, where $\|f\|_K = \sup\{|f(z)|: z \in K\}$ (see the proof of [4, p. 81, Theorem]). Also, for an inner function q, let $\operatorname{supp} q$ denote the support of q, that is, $\operatorname{supp} q$ is the set of all points on $\mathbf{T}$ for which there exists a sequence $\{z_n\}$ from the open unit disc such that $z_n \to z$ and $q(z_n) \to 0$. Thus, if a nonzero function f belongs to $qH^\infty \cap A$, then $f = 0$ on $\operatorname{supp} q$, so it follows that $\operatorname{supp} q$ is of measure zero (see [4, p. 52]) and $\bar{q}f$ is equal a.e. to a function in A. Also, the inner function q is analytic at each point on $\mathbf{T}$ which does not belong to $\operatorname{supp} q$. Therefore we have $qH^\infty \cap I(K) = qI(\operatorname{supp} q \cup K)$ for an inner function q and a closed subset K. It is known (see [4, p. 85, Theorem]) that J is a nonzero closed ideal of A if and only if $J = qI(K)$ where K is a closed subset of measure zero and q is an inner function such that $\operatorname{supp} q \subset K$.

Lemma 5. *Let J be a closed ideal of A and $J = qI(K)$, where K is a closed subset of measure zero and q is an inner function with* $\operatorname{supp} q \subset K$. *Then, for* *all* $f \in A$,

$$\begin{aligned}\|f + J\| &= \max\{\|f + qH^\infty\|, \|f\|_K\} \\ &= \max\{\|f + qH^\infty\|, \|f\|_{K\setminus \operatorname{supp} q}\}.\end{aligned}$$

Proof. Let $f \in A$ and take a measure μ on $\mathbf{T}$ annihilating $J = qI(K)$ such that $\|\mu\| = 1$ and

$$\|f + J\| = \int_{\mathbf{T}} f\, d\mu.$$

Since μ annihilates J, the proof of [4, p. 85, Theorem] shows that $d\mu = \bar{q}h\,dm + d\nu$ where $h \in zH^1$ and ν is a measure on $\mathbf{T}$ such that $\operatorname{supp}\nu \subset K$. Therefore we have

$$\begin{aligned}\|f + J\| &= \int_{\mathbf{T}} f\bar{q}h\, dm + \int_{\mathbf{T}} f\, d\nu \\ &\le \|f + qH^\infty\|\,\|h\|_1 + \|f\|_K\|\nu\| \\ &\le \max\{\|f + qH^\infty\|, \|f\|_K\}(\|h\|_1 + \|\nu\|) \\ &= \max\{\|f + qH^\infty\|, \|f\|_K\}.\end{aligned}$$

The converse inequality is obvious, so the first equality is proved. For the proof of the second equality, it suffices to show $\|f+qH^\infty\| \ge \|f\|_{\operatorname{supp} q}$. Take any $z \in \operatorname{supp} q$. Then there is a sequence $\{z_n\}$ from the open unit disc such that $z_n \to z$ and $q(z_n) \to 0$. Therefore, for all $h \in H^\infty$, $\|f+qh\| \ge |f(z_n)+q(z_n)h(z_n)| \to |f(z)|$, so that $\|f + qh\| \ge \|f\|_{\operatorname{supp} q}$. It follows that $\|f + qH^\infty\| \ge \|f\|_{\operatorname{supp} q}$.

Theorem 2. *If* Φ *is a representation of the disk algebra* A *on* H, *then* Φ *has a minimal* Q*-isometric dilation.*

Proof. Let $T = \Phi(z)$. First suppose that T is unitary. Then it follows from the spectral theory of unitary operators that $\ker\Phi = I(\operatorname{supp} T)$, where for a unitary operator U, $\operatorname{supp} U$ denotes the support of the spectral measure of U. (Note that $\operatorname{supp} T$ is of measure zero because $\Phi \ne 0$, and so T is singular.) We also have

$$\|\Phi(f)\| = \|f(T)\| = \|f\|_{\operatorname{supp} T} = \|f + I(\operatorname{supp} T)\|$$

for $f \in A$, hence Φ is Q-isometric.

Next suppose that T is not unitary. The contraction T is decomposed as $T = U \oplus T_1$ on $H = H_u \oplus H_1$, where U is unitary and T_1 is completely nonunitary. If T_1 is not of class C_0, then $\ker\Phi = \{0\}$. In this case we define $\widetilde{\Phi}\colon A \to L(H_u \oplus K)$ by $\widetilde{\Phi}(f) = f(U \oplus V)$, where V is the minimal isometric dilation on K of the contraction T_1. Since V has a unilateral shift summand, $\widetilde{\Phi}$ is isometric. It is easy to show that $\widetilde{\Phi}$ is a minimal Q-isometric dilation of Φ. If T_1 is of class C_0, then $\ker\Phi_{T_1} = qH^\infty$ where q is inner and

$$\ker\Phi = I_{\operatorname{supp} U} \cap qH^\infty = qI(K),$$

where $K = \operatorname{supp} U \cup \operatorname{supp} q$, which is of measure zero. By Lemma 4, there exists a contraction $\widetilde{T}_1$ on $\widetilde{H}_1$ satisfying the conditions (i), (ii) and (iii) in Lemma 4. Define $\widetilde{\Phi}\colon A \to L(H_u \oplus \widetilde{H}_1)$ by $\widetilde{\Phi}(f) = f(U \oplus \widetilde{T}_1)$. Then it easily follows from the conditions (i) and (iii) that $P_H\widetilde{\Phi}(f)|H = \Phi(f)$ for all $f \in A$ and $\bigvee_{f\in A}\widetilde{\Phi}(f)H = H_u \oplus \widetilde{H}_1$. For any $f \in A$, $\|\widetilde{\Phi}(f)\| = \max\{\|f\|_{\operatorname{supp} U}, \|f(\widetilde{T}_1)\|\}$ and $\|f(\widetilde{T}_1)\| = \|f + qH^\infty\|$ by the condition (ii) of $\widetilde{T}_1$, so it follows from

Lemma 5 that $\|\tilde{\Phi}(f)\| = \|f + \ker \Phi\|$. Thus $\tilde{\Phi}$ is a minimal Q-isometric dilation of Φ.

We are informed by the referee that the Ph.D. thesis of Che-Chen Chu, *Finite dimensional representation of a function algebra*, submitted to the University of Houston, 1992, contains the following stronger result of Theorem 1: If Φ: $A \to L(H)$ is a homomorphism and $\dim H = 2$, then the cb-norm of Φ is equal to the norm of Φ. However our proof of Theorem 1, which directly constructs the dilation, is different from Chu's proof.

REFERENCES

1. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York and Berlin, 1973.
2. B. Cole, K. Lewis, and J. Wermer, *Pick conditions on a uniform algebra and von Neumann inequalities*, J. Funct. Anal. **107** (1992), 235–254.
3. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, NJ, 1969.
4. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.
5. G. Misra, *Curvature inequalities and extremal properties of bundle shifts*, J. Operator Theory **11** (1984), 305–317.
6. V. Paulsen, *K-spectral values for some finite matrices*, J. Operator Theory **18** (1987), 249–263.
7. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.
8. B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060, JAPAN