PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

FORMULAS FOR THE JOINT SPECTRAL RADIUS OF NON-COMMUTING BANACH ALGEBRA ELEMENTS

P. ROSENTHAL AND A. SOLTYSIAK

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We discuss relations between several formulas for the joint spectral radius of n-tuples of non-commuting Banach algebra elements.

In 1960, Rota and Strang [10] proposed a formula for the joint spectral radius of an n-tuple (or, more generally, any bounded set) of Banach algebra elements. A different formula was investigated by Berger and Wang [3]. We consider relations between these formulas and the "geometric spectral radius".

Let $\mathscr{A}$ be a unital complex Banach algebra, and let $(a_1, \dots, a_n)$ be an n-tuple of elements of $\mathscr{A}$. Let $F(s, n)$ be the set of all functions from $\{1, \dots, s\}$ to $\{1, \dots, n\}$. Then the Rota-Strang radius $\hat{r}$ is defined by $\hat{r}(a_1, \dots, a_n) = \lim_{s\to\infty} \max_{f\in F(s,n)} \|a_{f(1)}a_{f(2)}\cdots a_{f(s)}\|^{1/s}$. (In [10] it is shown that this limit always exists.) The Berger-Wang radius r_* is defined by

$$r_*(a_1, \dots, a_n) = \limsup_{s\to\infty} \max_{f\in F(s,n)} [r(a_{f(1)}a_{f(2)}\cdots a_{f(s)})]^{1/s},$$

where r is the ordinary spectral radius in $\mathscr{A}$.

The left joint spectrum $\sigma_l(a_1, \dots, a_n)$ of an n-tuple of elements of $\mathscr{A}$ is the set of all n-tuples $(\lambda_1, \dots, \lambda_n)$ of complex numbers for which the left ideal generated by $\{a_1 - \lambda_1, a_2 - \lambda_2, \dots, a_n - \lambda_n\}$ is proper; the right joint spectrum is defined analogously. We denote by $\hat{\sigma}(a_1, \dots, a_n)$ the union of the left and right joint spectra of $(a_1, \dots, a_n)$, computed with respect to the unital algebra generated by $\{a_1, \dots, a_n\}$ (cf. [8]). The geometric joint spectral radius $r(a_1, \dots, a_n)$ of $(a_1, \dots, a_n)$ is then defined by

$$r(a_1, \dots, a_n) = \max\{|\lambda_j| : (\lambda_1, \dots, \lambda_n) \in \hat{\sigma}(a_1, \dots, a_n)\}.$$

That is, $r(a_1, \dots, a_n)$ is the radius of the smallest l^∞ ball in $\mathbb{C}^n$ containing $\hat{\sigma}(a_1, \dots, a_n)$.

Easy examples [8] show that $\hat{\sigma}(a_1, \dots, a_n)$ may be empty; in that case we put $r(a_1, \dots, a_n) = -\infty$.

Theorem 1. *For any* $(a_1, \dots, a_n)$,

$$r(a_1, \dots, a_n) \le \max_{1\le j\le n} r(a_j) \le r_*(a_1, \dots, a_n) \le \hat{r}(a_1, \dots, a_n).$$

Received by the editors January 10, 1994.
1991 *Mathematics Subject Classification.* Primary 46H05, 47A13.

Proof. The only inequality that is not immediate is the first. For this, assume that $(\lambda_1, \dots, \lambda_n) \in \hat{\sigma}(a_1, \dots, a_n)$. Then, by [5], there is a multiplicative linear functional ϕ on the unital Banach $\mathscr{A}_0$ algebra generated by $\{a_1, \dots, a_n\}$ such that $\phi(a_i) = \lambda_j$ for $j = 1, \dots, n$. Then $\phi(a_j)$ is in the spectrum of a_j computed with respect to the algebra $\mathscr{A}_0$. Since the ordinary spectral radius of an element does not depend on the algebra containing it, the result follows.

Example. There exists (a_1, a_2) with $r_*(a_1, a_2) = 0$ and $\hat{r}(a_1, a_2) = 1$.

Proof. We use an example constructed by Guinand [6] for a different purpose. Let $\mathscr{A}$ be the Banach algebra of all bounded linear operators on a separable complex Hilbert space. We construct two weighted shifts as follows. Let $\{\alpha_j\}$ be a sequence of 0's and 1's such that no finite substring occurs three times in a row; the existence of such a sequence was established by Thue ([12], Satz 6); also see ([7], Problem 1J). Choose an orthonormal basis $\{e_j\}_{j=0}^{\infty}$ and define

$$T_1 e_j = \alpha_j e_{j+1}, \qquad T_2 e_j = (1 - \alpha_j) e_{j+1}$$

for $j = 0, 1, 2, 3, \dots$. Then each T_i is a bounded linear operator, and every word in $\{T_1, T_2\}$ is nilpotent of order 3 (see [6]).

Since every word in $\{T_1, T_2\}$ is nilpotent, $r_*(T_1, T_2) = 0$. But there exist arbitrarily long words in $\{T_1, T_2\}$ of norm 1: simply let

$$f(j) = \begin{cases} 1 & \text{if } \alpha_{j-1} \neq 0, \\ 2 & \text{if } \alpha_{j-1} = 0 \end{cases}$$

for each j and form

$$W = T_{f(s)} T_{f(s-1)} \cdots T_{f(1)}.$$

Then $\|We_0\| = 1$, so $\|W\| = 1$. Thus $\max_{f \in F(s,2)} \|T_{f(1)} \cdots T_{f(n)}\|^{1/n} = 1$ for all s, and $\hat{r}(T_1, T_2) = 1$.

Notes. 1. For finite-dimensional matrices $\{T_i\}$, it is shown in [3] that

$$\hat{r}(T_1, \dots, T_n) = r_*(T_1, \dots, T_n).$$

2. Similarly, if the elements $\{a_i\}$ are mutually commuting, then it is shown in [11] that

$$r(a_1, \dots, a_n) = \hat{r}(a_1, \dots, a_n),$$

so by Theorem 1 it follows that $r_*(a_1, \dots, a_n) = \hat{r}(a_1, \dots, a_n)$ in that case too.

Let $\operatorname{rad}\mathscr{A}$ denote the Jacobson radical of $\mathscr{A}$.

Theorem 2. *If $\mathscr{A}$ is a unital complex Banach algebra, then the following are equivalent:*

(i) $\mathscr{A} / \operatorname{rad}\mathscr{A}$ *is commutative;*
(ii) $r(a_1, \dots, a_n) = \max\{r(a_j) : j = 1, \dots, n\}$ *for every finite subset* $\{a_1, \dots, a_n\}$ *of* $\mathscr{A}$;
(ii′) $r(a_1, a_2) = \max\{r(a_1), r(a_2)\}$ *for each pair* $\{a_1, a_2\} \subset \mathscr{A}$;
(iii) *for every* n, $r(a_1, \dots, a_n) = r_*(a_1, \dots, a_n)$ *for all n-tuples of elements of* $\mathscr{A}$;
(iii′) $r(a_1, a_2) = r_*(a_1, a_2)$ *for each pair* $\{a_1, a_2\} \subset \mathscr{A}$.

Proof. That (ii) implies (ii$'$) and that (iii) implies (iii$'$) are trivial. By Theorem 1, (iii) implies (ii) and (iii$'$) implies (ii$'$).

To show that (ii$'$) implies (i), let a and b be arbitrary elements of $\mathscr{A}$. By the definitions and the fact that the unital subalgebras generated by $\{a, b\}$ and $\{a+b, a\}$ are identical,

$$r(a+b, a) = \max\{|\alpha|: (\alpha, \beta) \text{ or } (\beta, \alpha) \text{ is in } \hat{\sigma}(a+b, a)\}.$$

Thus, in particular, $\hat{\sigma}(a+b, a) \neq \varnothing$. By [5], there is a unital multiplicative linear functional ϕ on the unital algebra generated by $\{a, b\}$ such that

$$r(a+b, a) = \max\{|\phi(a+b)|, |\phi(a)|\}.$$

Thus $r(a+b, a) \leq |\phi(a)| + |\phi(b)|$.

But clearly $|\phi(a)| \leq r(a)$ and $|\phi(b)| \leq r(b)$, since the spectral radius of a single element is independent of the Banach algebra containing it. Thus, $r(a+b, a) \leq r(a) + r(b)$. But $r(a+b) \leq r(a+b, a)$ by (ii$'$), so we have

$$r(a+b) \leq r(a) + r(b).$$

The Aupetit-Zemanek Theorem ([1], [13], or [2, p. 94]) states that such subadditivity of the spectral radius implies that $\mathscr{A}/\operatorname{rad}\mathscr{A}$ is commutative, so (ii$'$) implies (i).

It remains to be shown that (i) implies (iii). For an element $a \in \mathscr{A}$, let $[a]$ denote its coset in $\mathscr{A}/\operatorname{rad}\mathscr{A}$. Then $r(a) = r([a])$ for every $a \in \mathscr{A}$, so

$$r_*([a_1], \dots, [a_n]) = r_*(a_1, \dots, a_n).$$

But $\hat{\sigma}([a_1], [a_2], \dots, [a_n]) \subset \hat{\sigma}(a_1, \dots, a_n)$ (from the definitions), so

$$r([a_1], [a_2], \dots, [a_n]) \leq r(a_1, \dots, a_n).$$

Since $\mathscr{A}/\operatorname{rad}\mathscr{A}$ is commutative,

$$r([a_1], [a_2], \dots, [a_n]) = r_*([a_1], [a_2], \dots, [a_n]).$$

Thus

$$\begin{aligned} r_*(a_1, a_2, \dots, a_n) &= r_*([a_1], [a_2], \dots, [a_n]) \\ &= r([a_1], [a_2], \dots, [a_n]) \\ &\leq r(a_1, a_2, \dots, a_n). \end{aligned}$$

But, by Theorem 1, $r(a_1, a_2, \dots, a_n) \leq r_*(a_1, a_2, \dots, a_n)$, so (iii) holds.

Note. By Theorems 1 and 2, $r(a_1, a_2, \dots, a_n) = \hat{r}(a_1, a_2, \dots, a_n)$ implies each of the conditions of Theorem 2. We do not know, however, if the converse holds. Equivalently, if $\mathscr{A}$ is commutative modulo its radical, must

$$r_*(a_1, a_2, \dots, a_n) = \hat{r}(a_1, a_2, \dots, a_n)$$

for all n-tuples of elements of $\mathscr{A}$?

References

1. B. Aupetit, *Caractérisation spectrale des algébres de Banach commutatives*, Pacific J. Math. **63** (1976), 23–35.

2. ——, *A primer on spectral theory*, Universitext, Springer-Verlag, New York, 1991.

3. M. A. Berger and Y. Wang, *Bounded semigroups of matrices*, Linear Algebra Appl. **166** (1992), 21–27.

4. M. Chō and Żelazko, *On geometric spectral radius of commuting n-tuples of operators*, Hokkaido Math. J. **21** (1992), 251–258.

5. C. K. Fong and A. Sołtysiak, *Existence of a multiplicative linear functional and joint spectra*, Studia Math. **81** (1985), 213–220.

6. P. S. Guinand, *On quasinilpotent semigroups of operators*, Proc. Amer. Math. Soc. **86** (1982), 485–486.

7. P. R. Halmos, *Problems for mathematicians young and old*, Math. Assoc. Amer., Washington, DC, 1991.

8. R. E. Harte, *Spectral mapping theorems*, Proc. Roy. Irish Acad. Sect. A **72** (1972), 89–107.

9. V. Müller and A. Sołtysiak, *Spectrum of generators of a noncommutative Banach algebra*, Studia Math. **93** (1989), 87–95.

10. G.-C. Rota and W. G. Strang, *A note on the joint spectral radius*, Indag. Math. **22** (1960), 379–381.

11. A. Sołtysiak, *On the joint spectral radii of commuting Banach algebra elements*, Studia Math. **105** (1993), 93–99.

12. A. Thue, *Ueber unendliche Zeichenreihen*, Selected Mathematical Papers of Axel Thue, Universitetsforlaget, Oslo-Bergen-Tromsø, 1977, pp. 139–158.

13. J. Zemánek, *Spectral radius characterization of commutativity in Banach algebras*, Studia Math. **61** (1977), 257–268.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA M5S 1A1

E-mail address: `rosent@math.toronto.edu`

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, A. MICKIEWICZ UNIVERSITY, MATEJKI 48/49, 60–769 POZNAŃ, POLAND

E-mail address: `asoltys@plpuam11.bitnet`