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MORITA EQUIVALENCE OF TWISTED CROSSED PRODUCTS

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ABSTRACT. We introduce a natural notion of strong Morita equivalence of twisted actions of a locally compact group on C^*-algebras, and then show that the corresponding twisted crossed products are strongly Morita equivalent. This result is a generalization of the result of Curto, Muhly and Williams concerning strong Morita equivalence of crossed products by actions.

1. INTRODUCTION

The notion of strong Morita equivalence of C^*-algebras was introduced by Rieffel in his study of induced representations of C^*-algebras in [12]. Strong Morita equivalence plays an important role in the study of transformation group C^*-algebras (see [13] and [14]) and crossed products of C^*-algebras (see [5], [6], [9], [11] and [15]).

In this paper we discuss strong Morita equivalence of twisted crossed products. We introduce a natural notion of strong Morita equivalence of twisted actions which is sufficient to ensure strong Morita equivalence of the corresponding twisted crossed products. This result is a generalization of the result of Curto, Muhly, and Williams in [6, Theorem 1] and Combes in [5, §6] concerning strong Morita equivalence of crossed products by actions. In [3] we discuss strong Morita equivalence of crossed products by coactions and twisted crossed products by coactions, and in [4] we discuss strong Morita equivalence of crossed products by full coactions.

The main result of this paper is Theorem 2.3. Our purpose is to present a detailed proof which provides an imprimitivity bimodule at the level of spaces of functions and encompasses the case of [6, Theorem 1]. We also sketch another proof of Theorem 2.3 by applying [10, Theorem 3.4 and Corollary 3.7] and [6, Theorem 1].

2. MORITA EQUIVALENCE OF TWISTED CROSSED PRODUCTS

Definition 2.1. Let (A, G, α, u) and (B, G, β, v) be separable twisted dynamical systems in the sense of [10, Definition 2.1]. Suppose that X is a

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Banach A, B-imprimitivity bimodule. Let $\mathrm{Iso}(X)$ denote the set of all bijective linear isometries of X. An (α, u), (β, v)-compatible action of G on X is a map $\tau: G \to \mathrm{Iso}(X)$ satisfying the following conditions:

(i) for each $x \in X$, the map $s \mapsto \tau_s(x)$ from G into X is Borel;

(ii) ${}_A\langle\tau_s(x)|\tau_s(y)\rangle = \alpha_s({}_A\langle x|y\rangle)$, $\forall x, y \in X, \forall s \in G$,
$\langle\tau_s(y)|\tau_s(x)\rangle_B = \beta_s(\langle y|x\rangle_B)$, $\forall x, y \in X, \forall s \in G$;

(iii) $\tau_e(x) = x$, $\forall x \in X$,
$\tau_r(\tau_s(x)) = u(r, s)\tau_{rs}(x)v(r, s)^*$, $\forall x \in X, \forall r, s \in G$.

The twisted actions (α, u) and (β, v) are said to be strongly Morita equivalent by means of the imprimitivity system (X, τ).

Remark 2.2. We claim that this relation is an equivalence relation. It is clear that (α, u) is strongly Morita equivalent to itself by means of (A, α). Assume that (α, u) is strongly Morita equivalent to (β, v) by means of (X, τ). Let $\widetilde{X}$ be the opposite (also called the dual in [12, 6.17]) of the A, B-imprimitivity bimodule X, and put $\tilde{\tau}_s(\tilde{x}) = \widetilde{\tau_s(x)}$ for all $s \in G$ and $\tilde{x} \in \widetilde{X}$. Then $\tilde{\tau}$ is a (β, v), (α, u)-compatible action of G on $\widetilde{X}$. Finally, assume that (α, u) is strongly Morita equivalent to (β, v) by means of (X, τ^X) and (β, v) is strongly Morita equivalent to (γ, w) by means of (Y, τ^Y). Let $\widehat{\odot}_B$ be defined as in [3, §1] and put $Z = X\widehat{\otimes}_B Y$. As in [5, Remark 3.1], we define

$$\tau_s^Z(x\widehat{\odot}_B y) = \tau_s^X(x)\widehat{\odot}_B \tau_s^Y(y), \quad \forall x\widehat{\odot}_B y \in Z.$$

Since each τ_s^Z satisfies the second identity in (ii) of Definition 2.1, it is a linear isometry on $X\widehat{\odot}_B Y$ and extends to a linear isometry on the completion $Z = X\widehat{\otimes}_B Y$. Then τ^Z determines an (α, u), (γ, w)-compatible action of G on Z.

Let (A, G, α, u) be a separable twisted dynamical system. We will denote by $A \times_{\alpha,u} G$, or $\mathscr{A}$ for short, the twisted crossed product of this system (see [10, Definition 2.4]). The set $B_c(G; A)$ of (equivalence classes of) bounded measurable functions from G into A with compact support is a $*$-algebra with the convolution and the involution defined by

$$(f * g)(y) = \int f(x)[\alpha_x(g(x^{-1}y))]u(x, x^{-1}y)\,dx,$$
$$f^*(y) = \Delta_G(y)^{-1}u(y, y^{-1})^*[\alpha_y(f(y^{-1}))^*].$$

We denote this $*$-algebra by $B_c(A, G, \alpha, u)$ or $\mathscr{A}_c$ for short, and we will view it as a dense $*$-subalgebra of $A \times_{\alpha,u} G$.

Theorem 2.3. *Suppose that (A, G, α, u) and (B, G, β, v) are separable twisted dynamical systems. If the twisted actions (α, u) and (β, v) are strongly Morita equivalent by means of an imprimitivity system (X, τ), then $B_c(G, X)$ is a $B_c(A, G, \alpha, U)$, $B_c(B, G, \beta, v)$-imprimitivity bimodule. Therefore the twisted crossed products $A \times_{\alpha,u} G$ and $B \times_{\beta,v} G$ are strongly Morita equivalent.*

The proof of this theorem will follow from the next three lemmas.

First we establish the notation. Put

$$\mathscr{X}_c = B_c(G; X), \qquad \mathscr{Y}_c = B_c(G; \widetilde{X}).$$

We introduce what will be seen to be a right $\mathscr{B}_c$-rigged left $\mathscr{A}_c$-module structure on $\mathscr{X}_c$ as follows. For any $f \in \mathscr{A}_c$, ξ, $\eta \in \mathscr{X}_c$ and $g \in \mathscr{B}_c$, we define

$$\langle \eta|\xi\rangle_{\mathscr{B}_c}(s) = \int_G \langle \Delta_G(r^{-1})\tau_r(\eta(r^{-1}))v(r, r^{-1})|\tau_r(\xi(r^{-1}s))v(r, r^{-1}s)\rangle_B dr,$$

$$(\xi \cdot g)(s) = \int_G \xi(sr^{-1})\beta_{sr^{-1}}(g(r))v(sr^{-1}, r)\Delta_G(r^{-1})dr,$$

$$(f \cdot \xi)(s) = \int_G f(r)\tau_r(\xi(r^{-1}s))v(r, r^{-1}s)dr,$$

$$_{\mathscr{A}_c}\langle \xi|\eta\rangle(s) = \int_G {}_A\langle \xi(r)|\Delta_G(s^{-1}r)\tau_s(\eta(s^{-1}r))v(s, s^{-1}r)\rangle dr.$$

By [7, Proposition 15.15] the map $(r, s) \mapsto \tau_r(\xi(r^{-1}s))v(r, r^{-1}s)$ is measurable. Since the map $(x, y) \mapsto \langle x|y\rangle_B$ is continuous, it follows from [7, Corollary 9.13] that the map inside the first integral is also measurable. The measurability of the other maps can be proved in a similar way.

Similarly, we introduce what will be seen to be a right $\mathscr{A}_c$-rigged left $\mathscr{B}_c$-module structure on $\mathscr{Y}_c$ by replacing $\mathscr{A}_c$, α, u, $\mathscr{B}_c$, β, v and τ in the above formulas with $\mathscr{B}_c$, β, v, $\mathscr{A}_c$, α, u and $\tilde{\tau}$, respectively.

Lemma 2.4. *Let $T: \mathscr{X}_c \to \mathscr{Y}_c$ be defined by*

$$(T\xi)(s) = \Delta_G(s^{-1})[\tau_s(\xi(s^{-1}))v(s, s^{-1})]^{\sim}, \quad \forall \xi \in \mathscr{X}_c, \forall s \in G.$$

Then T is conjugate linear, and

(i) $T(f \cdot \xi) = (T\xi) \cdot f^*$, $\forall \xi \in \mathscr{X}_c, \forall f \in \mathscr{A}_c$;
$T(\xi \cdot g) = g^* \cdot (T\xi)$, $\forall \xi \in \mathscr{X}_c, \forall g \in \mathscr{B}_c$;

(ii) $\langle T\xi|T\eta\rangle_{\mathscr{A}_c} = {}_{\mathscr{A}_c}\langle \xi|\eta\rangle$, $\forall \xi, \eta \in \mathscr{X}_c$;
${}_{\mathscr{B}_c}\langle T\eta|T\xi\rangle = \langle \eta|\xi\rangle_{\mathscr{B}_c}$, $\forall \xi, \eta \in \mathscr{X}_c$.

Also T is bijective, and its inverse T^{-1} is given by

$$(T^{-1}\phi)(s) = \Delta_G(s^{-1})[\tilde{\tau}_s(\phi(s^{-1}))u(s, s^{-1})]^{\sim}, \quad \forall \phi \in \mathscr{Y}_c, \forall s \in G.$$

Proof. These assertions follow from routine computations. □

Lemma 2.5. *With the above notation, we have:*

(i) *For each $\xi \in \mathscr{X}_c$, $\langle \xi|\xi\rangle_{\mathscr{B}_c}$ is a positive element in $\mathscr{B}$.*

(ii) *The linear span of the range of $\langle\cdot|\cdot\rangle_{\mathscr{B}_c}$ is dense in $\mathscr{B}_c$.*

(iii) *For any $f \in \mathscr{A}_c$ and $\xi \in \mathscr{X}_c$, we have*

$$\langle f \cdot \xi|f \cdot \xi\rangle_{\mathscr{B}_c} \leq \|f\|^2_{\mathscr{A}} \langle \xi|\xi\rangle_{\mathscr{B}_c} \quad \text{in } \mathscr{B}.$$

Proof. (i) Let $(\pi, L, \mathscr{H})$ be a covariant representation of the twisted system (B, G, β, v) such that the integrated form $(\pi \times L, \mathscr{H})$ is faithful. For any $\eta, \eta' \in \mathscr{X}_c$ and $h, h' \in \mathscr{H}$, we have

$$(1) \qquad \langle (\pi \times L)(\langle \eta|\eta'\rangle_{\mathscr{B}_c})h|h'\rangle = \iint \langle \pi(\langle \eta(t)|\eta'(s)\rangle_B)L_s h|L_t h'\rangle \, ds\, dt.$$

Let $\eta = \sum_{i=1}^p \lambda_i \odot x_i \in B_c(G) \odot X$. By [16, Lemma IV.3.2] the matrix $\{\langle x_i|x_j\rangle_B\}$ is a positive element of $M_p(B)$, and therefore there is a matrix $\{b_{ij}\} \in M_p(B)$ such that

$$\langle x_i|x_j\rangle_B = \sum_{m=1}^p b^*_{mi} b_{mj}, \quad \forall i, j = 1, \dots, p.$$

We then obtain

$$(2)\qquad \langle \eta(t)|\eta(s)\rangle_B = \sum_{m=1}^{p}\left(\sum_{i=1}^{p}\lambda_i(t)b_{mi}\right)^*\left(\sum_{j=1}^{p}\lambda_j(s)b_{mj}\right).$$

It now follows from (1) and (2) that for any $h \in \mathscr{H}$

$$\langle(\pi\times L)(\langle\eta|\eta\rangle_{\mathscr{B}_c})h|h\rangle = \sum_{m=1}^{p}\left\|\int \pi\left(\sum_{j=1}^{p}\lambda_j(s)b_{mj}\right)L_s h\,ds\right\|.$$

Therefore $\langle\eta|\eta\rangle_{\mathscr{B}_c}$ is a positive element of $\mathscr{B}$. Since

$$\|\langle\xi|\xi\rangle_{\mathscr{B}_c} - \langle\eta|\eta\rangle_{\mathscr{B}_c}\|_{\mathscr{B}} \leq \|\xi\|_1\|\xi-\eta\|_1 + \|\xi-\eta\|_1\|\eta\|_1,$$

we deduce that $\langle\xi|\xi\rangle_{\mathscr{B}_c}$ is also a positive element of $\mathscr{B}$.

(ii) Suppose that the linear span I_c of the range of $\langle\cdot|\cdot\rangle_{\mathscr{B}_c}$ is not dense in $\mathscr{B}_c$. Let I be the closure of I_c, and let $(\pi, L, \mathscr{H})$ be a covariant representation of (B, G, β, v) such that $\ker(\pi\times L) = I$ and $\pi\times L \neq 0$. Since the linear span of elements $\langle x|x'\rangle_B$ with $x, x' \in X$ is dense in B, there are $\lambda\odot\langle x|x'\rangle_B \in B_c(G)\odot B$ and $h\in\mathscr{H}$ such that $\|(\pi\times L)(\lambda\odot\langle x|x'\rangle_B)h\|^2 \neq 0$. Put $\eta = \lambda\odot x\langle x|x'\rangle_B$ and $\xi = \lambda\odot x'$. We have

$$\|(\pi\times L)(\lambda\odot\langle x|x'\rangle_B)h\|^2 = \langle(\pi\times L)(\langle\eta|\xi\rangle_{\mathscr{B}_c})h|h\rangle.$$

Hence $(\pi\times L)(\langle\eta|\xi\rangle_{\mathscr{B}_c}) \neq 0$. This is a contradiction.

(iii) Let ω be a state of $\mathscr{B}$. Put

$$\langle\eta|\eta'\rangle_\omega = \omega(\langle\eta|\eta'\rangle_{\mathscr{B}_c}),\quad \forall\eta,\eta'\in\mathscr{X}_c.$$

Let $N_\omega = \{\eta\in\mathscr{X}_c\colon \langle\eta|\eta\rangle_\omega = 0\}$, let $q_\omega\colon \mathscr{X}_c \longrightarrow \mathscr{X}_c/N_\omega$ be the quotient map, and let $\mathscr{H}_\omega$ be the Hilbert space obtained by completing the space $\mathscr{X}_c/N_\omega$. The linear map $q_\omega\colon\mathscr{X}_c\longrightarrow\mathscr{H}_\omega$ is bounded with respect to the L^1-norm on $\mathscr{X}_c$. For any $a\in A$, $s\in G$ and $\eta\in\mathscr{X}_c$, we put

$$(\ell_A(a)\eta)(t) = a\eta(t),\qquad (\ell_G(s)\eta)(t) = \tau_s(\eta(s^{-1}t))v(s, s^{-1}t),$$

and define

$$\pi(a)(q_\omega(\eta)) = q_\omega(\ell_A(a)\eta),\qquad L_s(q_\omega(\eta)) = q_\omega(\ell_G(s)\eta).$$

Then $(\pi, L, \mathscr{H}_\omega)$ is a covariant representation of (A, G, α, u). Observe that

$$f\cdot\xi = \int \ell_A(f(s))\ell_G(s)\xi\,ds,$$

$$q_\omega(f\cdot\xi) = (\pi\times L)(f)q_\omega(\xi).$$

It then follows that

$$\begin{aligned}\omega(\langle f\cdot\xi|f\cdot\xi\rangle_{\mathscr{B}_c}) &= \|q_\omega(f\cdot\xi)\|^2 = \|(\pi\times L)(f)q_\omega(\xi)\|^2\\ &\leq \|f\|_{\mathscr{A}}^2\|q_\omega(\xi)\|^2 = \|f\|_{\mathscr{A}}^2\,\omega(\langle\xi|\xi\rangle_{\mathscr{B}_c}).\end{aligned}$$

Since this is true for all states of $\mathscr{B}$, the inequality in (iii) holds. □

Lemma 2.6. *With the above notation, we have:*

(i) *For each $\xi\in\mathscr{X}_c$, ${}_{\mathscr{A}_c}\langle\xi|\xi\rangle$ is a positive element in $\mathscr{A}$.*
(ii) *The linear span of the range of ${}_{\mathscr{A}_c}\langle\cdot|\cdot\rangle$ is dense in $\mathscr{A}_c$.*
(iii) *For any $g\in\mathscr{B}_c$ and $\xi\in\mathscr{X}_c$, we have*

$${}_{\mathscr{A}_c}\langle\xi\cdot g|\xi\cdot g\rangle \leq \|g\|_{\mathscr{B}}^2\,{}_{\mathscr{A}_c}\langle\xi|\xi\rangle \quad \textit{in } \mathscr{A}.$$

Proof. We first apply Lemma 2.5 to $\mathscr{Y}_c$ in place of $\mathscr{X}_c$ and $\langle\cdot|\cdot\rangle_{\mathscr{A}_c}$ in place of $\langle\cdot|\cdot\rangle_{\mathscr{B}_c}$. Then we use Lemma 2.4 to get the desired results. □

Remark 2.7. We now see that [6, Theorem 1] is a special case of Theorem 2.3. However Theorem 2.3 can also be proved by applying [10, Theorem 3.4 and Corollary 3.7] and [6 Theorem 1] as follows. Let $\mathscr{K}$ be the algebra of compact operators on the separable Hilbert space $L^2(G)$. Observe that $X\widehat{\otimes}\mathscr{K}$ is an $A\otimes\mathscr{K}$, $B\otimes\mathscr{K}$-imprimitivity bimodule. Since A and B are separable, so is X. Then we deduce from [2, Chapitre III, §3, Proposition 6 and Proposition 11] that $\mathrm{Iso}(X\widehat{\otimes}\mathscr{K})$ is a Polish group in the topology of pointwise norm convergence. We recall from [10, Theorem 3.4] that there are Borel maps $\mu\colon G\to UM(A\otimes\mathscr{K})$ and $\nu\colon G\to UM(B\otimes\mathscr{K})$, and there are strongly continuous actions ϕ of G on $A\otimes\mathscr{K}$ and ψ of G on $B\otimes\mathscr{K}$ such that for all $s\in G$,

$$\phi_s = \mathrm{Ad}\,\mu_s\circ(\alpha_s\otimes\mathrm{id}), \qquad \psi_s = \mathrm{Ad}\,\nu_s\circ(\beta_s\otimes\mathrm{id}).$$

For each $s\in G$, we put

$$T_s(\xi) = \mu_s(\tau_s\widehat{\otimes}\mathrm{id})(\xi)\nu_s^*, \quad \forall\xi\in X\widehat{\otimes}\mathscr{K}.$$

Then for any $r, s\in G$ and $\xi,\eta\in X\widehat{\otimes}\mathscr{K}$, we have

$$\begin{aligned}\phi_s({}_{A\otimes\mathscr{K}}\langle\xi|\eta\rangle) &= {}_{A\otimes\mathscr{K}}\langle T_s(\xi)|T_s(\eta)\rangle,\\ \psi_s(\langle\eta|\xi\rangle_{B\otimes\mathscr{K}}) &= \langle T_s(\eta)|T_s(\xi)\rangle_{B\otimes\mathscr{K}},\\ T_rT_s &= T_{rs},\end{aligned}$$

and each map $s\mapsto T_s(\xi)$ is Borel. Since $\mathrm{Iso}(X\widehat{\otimes}\mathscr{K})$ is a Polish group, the Borel homomorphism $T\colon G\to\mathrm{Iso}(X\widehat{\otimes}\mathscr{K})$ is continuous. Therefore ϕ, ψ and T satisfy the hypotheses of [6, Theorem 1], and hence $(A\otimes\mathscr{K})\times_\phi G$ and $(B\otimes\mathscr{K})\times_\psi G$ are strongly Morita equivalent. It then follows from [10, Corollary 3.7] that $(A\times_{\alpha,u}G)\otimes\mathscr{K}$ and $(B\times_{\beta,v}G)\otimes\mathscr{K}$ are strongly Morita equivalent, and therefore $A\times_{\alpha,u}G$ and $B\times_{\beta,v}G$ are strongly Morita equivalent.

Remark that by using [9, Proposition 5.1] and Theorem 2.3 we can show that if two Green's twisted actions are strongly Morita equivalent in the sense of [8, Definition 1], then so are the corresponding twisted crossed products. In [3] this notion of Morita equivalence was also reformulated into the context of coactions of a Hopf C^*-algebra on a Hilbert C^*-module in the sense of [1, Définition 2.2].

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