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ISOPARAMETRIC FUNCTIONS AND FLAT MINIMAL TORI IN $\mathbb{C}P^2$

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ABSTRACT. It is proved that all flat minimal tori in $\mathbb{C}P^2$ are unitarily congruent to the Clifford torus by studying a certain associated isoparametric function.

0. INTRODUCTION

A nonconstant smooth function f on a Riemannian manifold is said to be transnormal if $\|\nabla f\|^2 = a(f)$ for some smooth function a. If furthermore $\Delta F = b(f)$ for a smooth function b, then f is said to be isoparametric. Isoparametric functions on spheres have been the focus of extensive investigations in recent years; see, e.g., [1], [8], [9]. On the other hand, the fundamental paper [11] reveals that the transnormality of f alone implies certain important properties held by isoparametric functions.

In section one, we observe that the topological type of a compact Riemannian surface on which there is a transnormal function f is either a sphere (a projective plane if it is nonorientable), or a torus (a Klein bottle if nonorientable), and f is a function of the distance from an appropriate submanifold. As an immediate corollary, it follows that when the curvature of the Riemannian surface is constant, a transnormal function is also isoparametric; in the torus case, $\|\nabla f\|^2 = a(f)$ implies

$$\Delta f = 2^{-1}a'(f). \tag{0.1}$$

This identity will be important in section two.

Our look into transnormal functions on compact Riemannian surfaces is motivated by our study of compact minimal surfaces in $\mathbb{C}P^2$. Such minimal surfaces are divided into two classes, namely, the superminimal ones and the nonsuperminimal ones. Superminimal surfaces in $\mathbb{C}P^n$ in general are the projectivization of the vectors of Frenet frames of holomorphic curves in $\mathbb{C}P^n$, whereas nonsuperminimal surfaces in $\mathbb{C}P^2$ satisfy the following equations

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(known as Toda equations in soliton theory, which we will refer to henceforth):

$$\begin{aligned} \Delta \log p &= K + 2(q^2 + r^2 - 2p^2), \\ \Delta \log q &= K + 2(p^2 + r^2 - 2q^2), \\ \Delta \log r &= K + 2(p^2 + q^2 - 2r^2) \end{aligned} \tag{0.2}$$

for appropriate functions p, q and r (see section two), where $p^2 + q^2 = 1$, K is the curvature of the surface, and Δ is the surface Laplacian.

It is therefore of no surprise to learn that examples of compact nonsuperminimal surfaces are rare. Up to now, the only known examples of compact nonsuperminimal surfaces in $\mathbb{C}P^2$ have been the Clifford torus $[1: e^{ix}: e^{iy}]$, which is flat and totally real (i.e., $p = q$, or Kaehler angle $= \pi/2$), and the ones with S^1-symmetries. In fact, all totally real, flat minimal tori in $\mathbb{C}P^n$ have been classified in [6] (see also [7]), and the Clifford torus is the only one in $\mathbb{C}P^2$ up to unitary congruence.

It should be remarked that it is proved in [2] that all nonsuperminimal tori in $\mathbb{C}P^2$ arise from finite-type solutions in soliton theory, which provides a scheme, in principle, for the classification of all such surfaces.

Granted the difficulty in finding explicit examples of compact nonsuperminimal surfaces in $\mathbb{C}P^2$, the most natural starting place seems to be to classify those compact minimal surfaces of constant curvature. This restricts the genus of the surface to be ≥ 1. The classification in the constant Kaehler angle case has been carried out (cf. [5], [10]). Namely, the Clifford torus is the only compact constant Kaehler angle minimal surface in $\mathbb{C}P^2$ with genus ≥ 1.

The purpose of this paper is to classify all flat minimal tori in $\mathbb{C}P^2$.

Theorem. *All flat minimal tori in $\mathbb{C}P^2$ are unitarily congruent to the Clifford torus.*

It remains an interesting question to classify compact minimal surfaces of negative constant curvature in $\mathbb{C}P^2$.

The idea of the proof of the theorem is that on a flat minimal torus in $\mathbb{C}P^2$ the function r (and p and q as well) in the Toda equations (0.2) is an isoparametric function such that $\Delta r \neq 2^{-1}a'(r)$, where as before $\|\nabla r\|^2 = a(r)$, provided r is nonconstant. This contradicts (0.1), and so r must be a constant. Therefore the surface is necessarily totally real, which completes the proof.

1. Compact Riemannian surfaces with transnormal functions

Let M be a compact Riemannian manifold and let f be a transnormal function on M. Let $f(M) = [\alpha, \beta]$. We collect three properties of f proved in [11].

(I) The only critical values of f are α and β.

Set $V^+ = f^{-1}(\beta)$ and $V^- = f^{-1}(\alpha)$, called focal varieties.

(II) V^+ and V^- are smooth submanifolds (possibly with different dimensions on different connected components).

Let $c \in (\alpha, \beta)$. c is a regular value by (I). Since $\|\nabla f\|^2 = a(f)$ means any two level sets of f are parallel, set $r_\pm = d(f^{-1}(c), V^\pm)$ and consider the focal map $\varphi_\pm : f^{-1}(c) \to V^\pm$ given by $\varphi_\pm(p) = \exp_p(r_\pm \zeta_\pm)$, where ζ_+ (respectively, ζ_-) is the unit normal vector to $f^{-1}(c)$ pointing to the f-increasing (respectively, f-decreasing) direction.

(III) $\varphi_\pm\colon f^{-1}(c) \to V^\pm$ is a sphere bundle fibration.

We now specialize to the 2-dimensional case. By raising M to its double cover if necessary, we may assume without loss of generality that M is orientable; we will show that M is topologically either a sphere or a torus. The connected components of $V^\pm$ are now points and circles by (II). Pick a connected component V_β of V^+.

Case (i). V_β is a point set, say, $V_\beta = \{p_+\}$.

By (III), the geodesics emanating from p_+ will hit $f^{-1}(c)$ for all $c \in (\alpha, \beta)$. Since in a small neighborhood of p_+ the geodesic spheres around p_+ are diffeomorphic to S^1, we see that V_c, the geodesic sphere around p_+ at c induced by the Morse flow associated with the function f, is diffeomorphic to S^1 for all $c \in (\alpha, \beta)$. Let V_α be the connected component of V^- that V_c converges to as c approaches α.

If V_α is a point set $\{q_-\}$, then M is homeomorphic to S^2.

Otherwise V_α is a circle. By (III), $\varphi_-^{-1}(V_\alpha) \to V_\alpha$ is a sphere bundle whose fiber is S^0. Therefore, $\varphi_-^{-1}(V_\alpha)$ is a double cover of V_α. Clearly $\varphi_-^{-1}(V_\alpha) \supset V_c$. If $\varphi_-^{-1}(V_\alpha) = V_c$, then the tubular neighborhood $T = \bigcup V_t$, $t \in [c, \alpha]$, of V_α will be a Möbius band, which contradicts the orientability of M. Hence $\varphi_-^{-1}(V_\alpha) = V_c \cup W_c$ for some connected component W_c of $f^{-1}(c)$ other than V_c and $\partial T = V_c \cup W_c$. T is then a cylinder around V_α, and so T attached to the disc $\bigcup V_t$, $t \in [\beta, c]$, is still a disc. Now the Morse flow of f will carry W_c through certain connected level sets W_t to a connected component W_β of V^+. If W_β is a point set, M is homeomorphic to a sphere. Otherwise, the same reasoning as was done on V_α applied this time to W_β shows that the Morse flow will carry W_β through certain connected level sets X_t to a connected component X_α of V^-, so that the resulting space is either a sphere or a disc.

Continuing in this fashion, we see that the geodesics through p_+ expand geodesic discs until the last connected component of either V^+ or V^-, which must be a point for the disc to close up, is exhausted. M is thus homeomorphic to a sphere.

Case (ii). Every component of V^+ and V^- is homeomorphic to a circle.

Let $V_\beta \subset V^+$ be a connected component. As in Case (i), the tubular neighborhood T around V_β is a cylinder whose two boundary circles are carried via the Morse flow of f to two circles, etc. We see that eventually we will arrive at a cylinder whose two boundary circles are to be identified by a diffeomorphism of S^1. This gives us a torus.

From the construction, we see that in the sphere case the polar coordinates (γ, θ) around p_+, and in the torus case the cylindrical coordinates (γ, θ), where θ is the arc length of the starting circle $V_\beta \subset V^+$ and γ is the geodesic length of geodesics perpendicular to V_β, give a coordinate chart of M on which the transnormal function f is a function of γ alone. Hence the Riemannian metric is $ds^2 = d\gamma^2 + G(\gamma, \theta)d\theta^2$, where $J = G^{1/2}$ satisfies the Jacobi equation $\partial^2 J/\partial\gamma^2 + KJ = 0$ with initial conditions $J(0, \theta) = 0$ and $\partial J/\partial\gamma(0, \theta) = 1$ in the sphere case, and $J(0, \theta) = 1$ and $\partial J/\partial\gamma(0, \theta) = -\kappa(\theta)$, where κ is the geodesic curvature of the curve V_β, in the torus case.

Observe that

$$f'(\gamma)^2 = \|\nabla f\|^2 = a(f). \tag{1.1}$$

In particular, when K is a positive constant, $G^{1/2} = K^{-1/2}\sin K^{1/2}\gamma$, and

$$\begin{aligned}\Delta f &= Kf''(\gamma)^2 + Kf'(\gamma)\cot(\gamma)\\ &= 2^{-1}Ka'(f) + Ka(f)^{1/2}\cot(\gamma),\end{aligned}$$

where $d\gamma/df = a(f)^{-1/2}$. It follows that the transnormal function is in fact isoparametric.

When $K = 0$, we lift M to its universal cover. V_β and its translates via the Morse flow of f will be lifted up to curves V_t^* diffeomorphic to a line, whereas the geodesics perpendicular to V_β will be lifted to lines which are perpendicular to the family V_t^*. Any two of these lines never intersect since they do not on M; hence these lines are parallel lines. Being perpendicular to these parallel lines, V_t^* must therefore be parallel lines too. Back on M this says that $\kappa = 0$, and so $G \equiv 1$. Differentiating (1.1) and noting that $\Delta f = f''(r)$ in this case gives

$$\Delta f = 2^{-1}a'(f), \tag{1.2}$$

so that again the transnormal function f is isoparametric. (1.2) is to be employed in the next section.

2. Flat minimal tori in $\mathbb{C}P^2$

Let M be a compact Riemann surface and let $\mathbb{C}P^n$ be equipped with the Fubini-Study metric $\langle\ ,\ \rangle_{\mathbb{C}P^n}$ whose curvature is normalized to be 4. Fix a metric ds^2 in the conformal class of M and let $f_0\colon (M, ds^2) \to \mathbb{C}P^n$ be a branched minimal immersion, i.e., $f_0^*\langle\ ,\ \rangle_{\mathbb{C}P^n} = \lambda\, ds^2$ for some nonnegative function λ and $\mathrm{tr}(\nabla df_0) = 0$. Denote by $\mathfrak{L}$ the tautological bundle over $\mathbb{C}P^n$. Then $L = f_0^{-1}\mathscr{L}$ inherits a natural holomorphic bundle structure from those of M and $\mathbb{C}P^n$, and so does $L^\perp$, the hyperplane bundle perpendicular to L in $M \times \mathbb{C}^{n+1}$ (cf. [4]). For a local coordinate z, the Gram-Schmidt process defines a map G_z from L to $L^\perp$ given by $G_z(X) = \partial X/\partial z - \{\langle \partial X/\partial z, X\rangle/\|X\|\}X$, where $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbb{C}^{n+1}$. Then conformality and harmonicity of f_0 imply the following:

(1) The well-defined map $f_1(p) = G_z(L_p)$ from $p \in M$ to $\mathbb{C}P^n$ is conformal and harmonic. Denote f_1 by ∂f_0.

(2) The map $\partial(X) = G_z(X) \otimes dz$ from $X \in L$ to $L^\perp \otimes T^{(1,0)}M$ is a well-defined holomorphic bundle map.

Clearly the procedures (1) and (2) can be successively carried on so that one obtains $f_0 \to f_1 \to f_2 \to \cdots$. One sets $L_i = f_i^{-1}\mathscr{L}$. Similarly one can define $f_{\overline{1}}, f_{\overline{2}}, \ldots$ and $\overline{L}_i$ by replacing $\partial/\partial z$ by $\partial/\partial \overline{z}$ in (1) and (2). Conformality of f_0 implies that L_0, L_1, $L_{\overline{1}}$ are mutually orthogonal.

In particular if $n = 2$ and if f_0 is neither holomorphic nor anti-holomorphic, then we have either of the following:

(a) $0 = f_2 = \partial f_1$ (respectively, $f_{\overline{2}} = 0$). It follows that $f_0 = \overline{\partial} f_1$ (respectively, $= \partial f_{\overline{1}}$) and f_1 (respectively, $f_{\overline{1}}$) is anti-holomorphic (respectively, holomorphic). f_0 is said to be superminimal.

(b) $f_{\overline{1}} = f_2$, so that the ∂-process is cyclic. f_0 is said to be nonsuperminimal.

Quantitatively, pick orthonormal vectors Z_0, Z_1, Z_2 spanning L_0, L_1, L_2, respectively. Let $\varphi = \theta_1 + \sqrt{-1}\theta_2$ be a complex coframe on M. Then $d\varphi =$

$\sqrt{-1}\omega\wedge\varphi$ with ω the connection form. Then (b) says (cf. [3])

$$\begin{aligned} dZ_0 &= \Psi_0 Z_0 + s\varphi Z_1 + \overline{t\varphi} Z_2\,,\\ dZ_1 &= -\overline{s\varphi} Z_0 + \Psi_1 Z_1 + c\varphi Z_2\,,\\ dZ_2 &= -t\varphi Z_0 - \overline{c\varphi} Z_1 + \Psi_2 Z_2\,, \end{aligned} \tag{2.1}$$

where Ψ_0, Ψ_1, Ψ_2 are the connection forms of the bundles L_0, L_1, $L_2 = L_{\overline{1}}$. Note that f_0 is superminimal precisely when $c \equiv 0$. Furthermore, the holomorphy of the map $\partial\colon L_0 \to L_1 \otimes T^{(1,0)}M$, i.e., $\partial(Z_0) = sZ_1 \otimes \varphi$, gives that the difference between the first Chern classes of $L_1 \otimes T^{(1,0)}M$ and L_0 is the ramification index of ∂, or expressed analytically (cf. [5]),

$$\Delta(\log|s|)\varphi\wedge\overline{\varphi} = K\varphi\wedge\overline{\varphi} + 2d(\Psi_0 - \Psi_1)\,,$$

or

$$\Delta(\log|s|) = K + 2(|t|^2 + |c|^2 - 2|s|^2).$$

Similarly by considering $\partial\colon L_1 \to L_2 \otimes T^{(1,0)}M$ with $\partial(Z_1) = cZ_2 \otimes \varphi$, and $\partial\colon L_2 \to L_0 \otimes T^{(1,0)}M$ with $\partial(Z_2) = -tZ_0 \otimes \varphi$, one deduces the Toda equations (0.2) when one sets $p = |s|$, $q = |t|$ and $r = |c|$. From (2.1) it follows that $p^2 + q^2 = 1$.

Note that adding the three equations in (0.2) results in $\Delta(\log pqr) = 3K$, which implies that when M is a torus p, q and r have no zeros by the Gauss-Bonnet theorem.

Assume now that $K = 0$. Then $\Delta(\log pqr) = 3K = 0$, and so pqr is a constant. Set $pqr = \delta$ and $r = e^u$. We obtain the next crucial lemma.

Lemma. $\|\nabla u\|^2 = -\delta^{-2}e^{4u} + 2(\delta^{-2} + 2)e^{2u} - 14 + 24\delta^2 e^{-2u}$,

$$\Delta u = 2 - 4e^{2u}.$$

Proof. Recall the general formula $\Delta(\log h) = h^{-2}[h\Delta h - \|\nabla h\|^2]$ and $p^2 + q^2 = 1$. Then the first two formulas of (0.2) can be expanded as

$$\begin{aligned} p^2\Delta p^2 - \|\nabla p^2\|^2 &= 4(1 + r^2 - 3p^2)p^4q^2\,,\\ q^2\Delta q^2 - \|\nabla q^2\|^2 &= 4(1 + r^2 - 3q^2)p^2q^4. \end{aligned}$$

Knowing that $\|\nabla p^2\| = \|\nabla q^2\|$ and $\Delta p^2 = -\Delta q^2$, we can solve the two equations for $\|\nabla p^2\|^2$. Namely,

$$\|\nabla p^2\|^2 = -p^2q^2(4r^2 - 8 + 24p^2q^2).$$

Differentiating $p^2q^2r^2 = \delta^2$ with $\nabla p^2 = -\nabla q^2$ and $\nabla r^2 = 2r^2\nabla u$ in mind, we have

$$2\delta^2\nabla u = r^2(p^2 - q^2)\nabla p^2.$$

Taking the norm on both sides of the equation and employing $p^2 + q^2 = 1$, $p^2q^2r^2 = \delta^2$ and $r^2 = e^{2u}$, we arrive at the first equation of the lemma. The second equation of the lemma is an immediate substitution of $r = e^u$ and $p^2 + q^2 = 1$ into the third equation of (0.2).

We are ready to draw the conclusion of the theorem we aim to prove.

Theorem. *All flat minimal tori in $\mathbb{C}P^2$ are unitarily congruent to the Clifford torus.*

Proof. Notation is as in the preceding lemma. Suppose u is not a constant. Then u is an isoparametric function by the lemma. It is immediate to see that

$\Delta u \neq 2^{-1}a'(u)$, where as before $a(u)$ stands for the function on the right-hand side of the identity for $\|\nabla u\|^2$ in the lemma. This contradicts (1.2). Therefore u is a constant, and so is $r = e^u$. Now $p^2q^2r^2 = \delta^2$ and $p^2 + q^2 = 1$ give the constancy of all p, q and r. It then follows from (0.2) that $p = q = r$, which characterizes the Clifford torus up to unitary equivalence.

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