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A SHORT NOTE ON THE FULL JACOBI GROUP

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ABSTRACT. The full Jacobi group has been defined to study Jacobi forms. The full Jacobi group has properties similar to the modular group. In this note we investigate properties of the full Jacobi group to get generators and relations.

1. INTRODUCTION

Recently, there have been many studies on the Jacobi forms. Jacobi forms are a mixture of modular forms and elliptic functions. Even though the classical examples are given as the Jacobi theta functions and the Fourier coefficients of Siegel modular forms of genus two, systematic studies have been developed only recently. They are closely related to half integral weight modular form, period of modular form, Heegner points, etc. [see, for example, [1]).

The full Jacobi group has been defined to study Jacobi forms. The full Jacobi group has properties similar to the modular group. In this note we investigate properties of the full Jacobi group to get generators and relations. The relations among the elements in the group will be a useful tool for studying the period of Jacobi forms.

2. DEFINITIONS AND THEOREM

Let $\Gamma(1)$ be a modular group, that is, a set of 2 by 2 matrices with integer entries and the determinant 1. The Jacobi group is defined as followings;

Definition 2.1. $\Gamma(1)^J := \Gamma(1) \propto Z^2 = \{[M, X] \mid M \in \Gamma(1), X \in Z^2\}$. This set $\Gamma(1)^J$ forms a group under a group law $[M, X][M', X'] = [MM', XM' + X']$, and is called the full Jacobi group.

Here, we use the following notation.

Notation 2.1.

$$(1) \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad P = -TS, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

First of all we state the following well-known lemma without proof (see [3], [4]).

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Lemma 2.1. (1) $\Gamma(1)$ *is generated by* S *and* T. *Every element* $M \in \Gamma(1)$ *can be written in the form*

$$M = S^{q_0} T S^{q_1} T \cdots T S^{q_n}, \quad \textit{for } q_i \in Z \ (0 \le i \le n). \tag{2}$$

(2) *Since* S *and* T *generate the group* $\Gamma(1)$, *so do* T *and* $P = -TS$, *for* $S = TP$.

(3) *The generators* T *and* P *of the group* $\Gamma(1)$ *satisfy the relations*

$$T^4 = P^3 = I, \qquad T^2 P = P T^2, \tag{3}$$

and these are defining relations for $\Gamma(1)$.

We have a similar result on the full Jacobi group. We use the following notation.

Notation 2.2.

$$\begin{gathered} G_0 = [S, (0, 0)], \qquad G_1 = [S, (1, 0)], \qquad G_2 = [T, (1, 0)], \\ G_3 = [I, (1, 0)], \qquad G_4 = [I, (0, 1)], \qquad I^J = [I, (0, 0)], \\ V = G_2^3 G_1 = [-TS, (1, -1)], \qquad R = G_2^3 G_0 = [-TS, (0, -1)]. \end{gathered} \tag{4}$$

Now we state the main theorem.

Theorem 2.1. (1) $\Gamma(1)^J$ *is generated by* G_1 *and* G_2; *every element* $[M, (\lambda, \mu)] \in \Gamma(1)^J$ *can be written in the form*

$$[M, (\lambda, \mu)] = G_1^{q_0} G_2 G_1^{q_1} G_2 \cdots G_2 G_1^{q_n}, \tag{5}$$

where $q_i \in Z \ (0 \le i \le n)$.

(2) $\Gamma(1)^J$ *is generated by* G_0 *and* G_2. $[M, (\lambda, \mu)] \in \Gamma(1)^J$ *can be written in the form*

$$[M, (\lambda, \mu)] = G_0^{q_0} G_2 G_0^{q_1} G_2 \cdots G_2 G_0^{q_n}, \tag{6}$$

where $q_i \in Z \ (0 \le i \le n)$.

(3) $\Gamma(1)^J$ *is generated by* G_2 *and* V. *The generators* G_2 *and* V *of the group* $\Gamma(1)^J$ *satisfy the relations*

$$\begin{gathered} G_2^4 = V^3 = I^J, \\ V G_2^2 = [I, (-1, -2)] G_2^2 V = G_2^2 [I, (1, 2)] V = G_2^2 V [I, (-2, -1)], \end{gathered} \tag{7}$$

and these are defining relations for $\Gamma(1)^J$.

(4) $\Gamma(1)^J$ *is generated by* G_2 *and* R.

Proof of Theorem 2.1. (1) Let Γ^J_{sub} be a subgroup generated by G_1, G_2, G_3, and G_4. We first claim that $\Gamma^J_{\mathrm{sub}} = \Gamma(1)^J$; it is clear that $\Gamma^J_{\mathrm{sub}} \subset \Gamma(1)^J$. To prove the other direction we have the following from Lemma 2.1:

$$G_1^{q_0} G_2 G_1^{q_1} G_2 \cdots G_2 G_1^{q_n} = [M, (\lambda', \mu')], \quad \text{for } (\lambda', \mu') \in Z^2. \tag{8}$$

Therefore, for any $[M, (\lambda, \mu)] \in \Gamma(1)^J$, there exist integers $\alpha, \beta \in Z$ such that

$$[M, (\lambda, \mu)] = G_1^{q_0} G_2 G_1^{q_1} G_2 \cdots G_2 G_1^{q_n} [I, (\alpha, \beta)] = [M, (\lambda', \mu')] G_3^{\alpha} G_4^{\beta}. \tag{9}$$

So we checked that $\Gamma^J_{\mathrm{sub}} = \Gamma(1)^J$.

Next we show that $\Gamma^J_{\text{sub}} = \Gamma^J$, where Γ^J is generated only by G_1 and G_2. We claim that $G_3, G_4 \in \Gamma^J$. It is easy to check that

$$\begin{aligned} &G_4 = [I, (0, 1)] = (G_2G_1)^2G_2^3(G_1G_2^2)(G_1G_2)^3, \\ &G_3 = G_2^3G_4G_2. \end{aligned} \tag{10}$$

(2) It is sufficient to show that G_0 and G_2 generate G_1 and G_2. Since $G_2^2G_0 = [-S, (1, 0)]$ and $G_2^2(G_0G_2)^3G_2^2 = [-I, (0, 0)]$, we get $G_1 = [S, (1, 0)] = G_2^2(G_0G_2)^3G_0$. We also note that $G_0^{-1}G_2^2(G_0G_2)^3G_0 = G_3$. The remaining proof is similar to that of Theorem 2.1(1).

(3) Since G_1 and G_2 generate the group $\Gamma(1)^J$, so do G_2 and V, for $G_1 = G_2V$. Furthermore, we can easily check that G_2 and V satisfy the relations (7). Suppose now that an arbitrary relation is given which we assume without restriction to have the form

$$G_2^{\varepsilon_1}V^{\delta_1}G_2^{\varepsilon_2}V^{\delta_2}\cdots G_2^{\varepsilon_n}V^{\delta_n}G_2^{\varepsilon_{n+1}} = I^J, \quad \text{where } \varepsilon_i, \delta_j \in Z. \tag{11}$$

By applying (7), we can find the unique pair of integers (α, β) such that

$$G_2^{\varepsilon_1}[I, (\alpha, \beta)]V^{\delta_1}G_2V^{\delta_2}G_2\cdots G_2V^{\delta_n}G_2^{\varepsilon_n} = I^J, \tag{12}$$

where $\varepsilon_1, \varepsilon_{n+1} = 0, 1, 2$, or 3, $\delta_i = 1$ or 2, and $n \geq 0$.

After a suitable multiplication by G_2 we obtain

$$[I, (\alpha, \beta)]V^{\delta_1}G_2V^{\delta_2}G_2\cdots G_2V^{\delta_n} = G_2^{\delta}, \tag{13}$$

where $\delta_i = 1$ or 2, $\delta = 0, 1, 2$, or 3, and $n \geq 0$. Then this representation becomes

$$\begin{aligned} &[I, (\alpha, \beta)]V^{\delta_1}G_2V^{\delta_2}G_2\cdots G_2V^{\delta_n} \\ &\quad = [P^{\delta_1}TP^{\delta_2}T\cdots TP^{\delta_n}, (\alpha', \beta')] \\ &\quad = [T^{\delta}, (\alpha'', \beta'')], \quad \text{for some } \alpha', \beta', \alpha'', \beta'' \in Z. \end{aligned} \tag{14}$$

So we have that

$$P^{\delta_1}TP^{\delta_2}T\cdots TP^{\delta_n} = T^{\delta}, \tag{15}$$

where $\delta_i = 1$ or 2, $\delta = 0, 1, 2$, or 3, and $n \geq 0$. However, this is not possible for $n \geq 1$ by Lemma 2.1(3). So equation (7) is the only relation in the group $\Gamma(1)^J$.

(4) Since $G_0 = G_2R$, the group $\Gamma(1)^J$ is generated by G_2 and R.

From the above theorem, we get the following corollary.

Corollary 2.1. (1) *The generators* G_1 *and* G_2 *of the group* $\Gamma(1)^J$ *satisfy the relations*

$$\begin{aligned} &G_2^4 = (G_2^3G_1)^3 = I^J, \\ &G_1G_2^2 = [I, (2, -1)]G_2^2G_1 = G_2^2[I, (-2, 1)]G_1 \\ &\qquad = G_2^2G_1[I, (-2, -1)] \end{aligned} \tag{16}$$

and these are defining relations for $\Gamma(1)^J$.

(2) *The generators* G_0 *and* G_2 *of the group* $\Gamma(1)^J$ *satisfy the relations*

$$\begin{aligned} G_2^4 &= (G_2^3 G_0)^3 = I^J , \\ G_0 G_2^2 &= [I , (0 , 1)] G_2^2 G_0 = G_2^2 [I , (0 , -1)] G_0 \\ &= G_2^2 G_0 [I , (0 , -1)] \end{aligned} \tag{17}$$

and these are defining relations for $\Gamma(1)^J$.

(3) *The generators* R *and* G_2 *of the group* $\Gamma(1)^J$ *satisfy the relations*

$$\begin{aligned} G_2^4 &= R^3 = I^J , \\ R G_2^2 &= [I , (1 , 0)] G_2^2 R = G_2^2 [I , (-1 , 0)] R = G_2^2 R [I , (0 , -1)] \end{aligned} \tag{18}$$

and these are defining relations for $\Gamma(1)^J$.

Proof of Corollary 2.1. (1) Since $V = G_2^3 G_1$ and $V^3 = I^J$, we get $(G_2^3 G_1)^3 = I^J$. Also, $G_1 G_2^2 = G_2 V G_2^2 = G_2 [I , (-1 , -2)] G_2^2 V = G_2 [I (-1 , -2)] G_2^3 G_2^2 G_1 = [I , (2 , -1)] G_2^2 G_1$.

(2) Since $G_0 = G_2^3 V G_2^3 (V G_2^2)^3 G_2 = G_1 [I , (-1 , 0)] = [I , (-1 , 1)] G_1$, by applying relations on (16), we get (17).

(3) Since $R = G_2^3 G_0$, we get (18) by applying (17).

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