

RELATIONS AMONG HOMOTOPY OPERATIONS FOR SIMPLICIAL COMMUTATIVE ALGEBRAS

PAUL G. GOERSS AND THOMAS J. LADA

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ABSTRACT. The homotopy groups of a simplicial commutative algebra over the field with two elements support natural operations. Here we write the relations among these operations in an admissible form.

Let A be a simplicial commutative algebra over the field \mathbb{F}_2 . Then A is, among other things, a simplicial set and, as such, has homotopy groups. In fact,

$$\pi_n A \cong H_n(A, \partial)$$

where $\partial = \sum d_i: A_n \rightarrow A_{n-1}$ is the sum (or alternating sum) of the face operators. By the Eilenberg-Zilber Theorem, $\pi_* A$ is a graded commutative \mathbb{F}_2 algebra. Cartan [2], Bousfield [1], and Dwyer [3] have pointed out the existence of natural operations on these homotopy groups. Indeed, Dwyer proved the following theorem.

Theorem 1. *Let A be a simplicial commutative \mathbb{F}_2 algebra. Then there are natural operations*

$$\delta_i: \pi_n A \rightarrow \pi_{n+i} A, \quad 2 \leq i \leq n,$$

so that

- (1) $\delta_i(x + y) = \delta_i(x) + \delta_i(y)$ if $i < n$ and $\delta_n(x + y) = \delta_n(x) + \delta_n(y) + xy$;
- (2) $\delta_i(xy) = 0$ unless $x \in \pi_0 A$ or $y \in \pi_0 A$. If $x \in \pi_0 A$, then $\delta_i(xy) = x^2 \delta_i(y)$; if $y \in \pi_0 A$, then $\delta_i(xy) = \delta_i(x)y^2$;
- (3) if $j < 2i$, then there is a relation

$$\delta_j \delta_i(x) = \sum_{(j+1)/2 \leq k < i} \binom{i-j+k-1}{i-k} \delta_{i+j-k} \delta_k(x).$$

Dwyer further points out that if $x \in \pi_n A$, $n \geq 1$, then $x^2 = 0$.

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The purpose of this note is to discuss the relations of Theorem 1, part (3). It is clear that any composition $\delta_j \delta_i$ can be rewritten as a linear combination of the compositions $\delta_q \delta_p$ which are admissible in the sense that $q \geq 2p$. However, the relations as written do not immediately return such a sum but have to be applied recursively. Our main point is to get a closed formula, which is supplied in the next result.

Proposition 2. *Let A be a simplicial commutative \mathbb{F}_2 algebra and $x \in \pi_n A$. Then if $j < 2i$ there is a relation*

$$\delta_j \delta_i(x) = \sum_{(j+1)/2 \leq k \leq (i+j)/3} \binom{i-j+k-1}{i-k} \delta_{i+j-k} \delta_k(x).$$

Notice that the binomial coefficients haven't changed (an amusing curiosity) but the limits on k have; indeed, since $k \leq \frac{i+j}{3}$ one has $i+j-k \geq 2k$ and the relations now return a sum of admissible compositions.

This result has been known to the second author for some time (see [5]) and was proved independently, but not published, by the first author for the work in [4].

The motivation for studying these relations was this: in the work of the first author on the cohomology of commutative \mathbb{F}_2 algebras [4], it is fairly simple to construct certain primary cohomology operations analogous to Steenrod operations. Then one wants to know the "Adem relations" among them. These are obtained by a "Koszul duality" argument (cf. [6]) once one has the relations among the operations δ_i in closed form, as in Proposition 2. And, in fact, it was low-degree calculations in the cohomology of commutative \mathbb{F}_2 algebras that suggested the truth of Proposition 2.

The proof of Proposition 2 is modeled on one of the standard ways of producing the Adem relations in the Steenrod algebra. If \mathcal{A} is the Steenrod algebra at the prime 2, then there is a derivation $\partial: \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\partial(\text{Sq}^i) = \text{Sq}^{i-1}.$$

The Adem relations, then, are consequences of the existence of this derivation and the relations

$$\text{Sq}^{2n-1} \text{Sq}^n = 0, \quad n \geq 1.$$

A similar argument will apply here.

We form an algebra analogous to the Steenrod algebra out of the operations δ_i . Let \mathcal{D} be the graded \mathbb{F}_2 algebra formed by taking the quotient of the tensor algebra on symbols δ_i of degree i , $i \geq 2$, by the ideal generated by the relations, for $j < 2i$

$$R(j, i) = \sum_{(j+1)/2 \leq k \leq i} \binom{i-j+k-1}{i-k} \delta_{i+j-k} \delta_k.$$

These are the relations implicit in Theorem 1, part (3).

The algebra \mathcal{D} comes equipped with a degree-raising derivation $\partial: \mathcal{D} \rightarrow \mathcal{D}$. In fact, ∂ is defined by the requirement that it be a derivation over \mathbb{F}_2 and the formula

$$\partial(\delta_i) = \delta_{i+1}.$$

That this function is well defined is implied by the following result.

Lemma 3. *We have that*

$$\partial R(j, i) = R(j + 1, i) + R(j, i + 1)$$

where, if $j + 1 = 2i$, $R(j + 1, i) = 0$.

Proof. If $j + 1 < 2i$, this is a routine, if tedious, exercise in binomial coefficients. Because of the denominator in the limits of the summation, it is easiest to break the calculation into two cases: one where j is even and one where j is odd. We omit the details. If $j + 1 = 2i$, one simply checks that

$$\begin{aligned} R(2i - 1, i) &= \delta_{2i-1}\delta_i, \\ R(2i - 1, i + 1) &= \delta_{2i}\delta_i + \delta_{2i-1}\delta_{i+1} \end{aligned}$$

as required. \square

We next restate Proposition 2 in a form suitable for induction.

Proposition 4. *Let $t > 0$ and $2b - t \geq 2$. Then in the algebra \mathcal{D}*

$$\delta_{2b-t}\delta_b = \sum_{t/3 \leq s \leq (t-1)/2} \binom{t-s-1}{s} \delta_{2b-t+s}\delta_{b-s}.$$

Notice that Proposition 4 implies Proposition 2 by setting $b = i$, $t = 2i - j$, and $s = b - k = i - k$.

The rest of this note is devoted to the proof of Proposition 4. This is accomplished by induction on t . The only point obscuring the argument is that because of the 2 and 3 in the denominators of the summation limits in the formula, there are six cases, one for each congruence class of t modulo 6.

To begin with, note that if $t = 1$ or $t = 2$, the formula in Proposition 5 merely restates $R(2b-1, b)$ or $R(2b-2, b)$ respectively. The cases $t = 3, 4, 5$ and 6 are covered using the derivation ∂ as follows.

The case $t = 3$: One has

$$0 = \partial(\delta_{2a-1}\delta_a) = \delta_{2a}\delta_a + \delta_{2a-1}\delta_{a+1}.$$

Setting $a = b - 1$ yields the desired formula.

The case $t = 4$: One has

$$0 = \partial(\delta_{2a-2}\delta_a) = \delta_{2a-1}\delta_a + \delta_{2a-2}\delta_{a+1} = \delta_{2a-2}\delta_{a+1}.$$

Setting $a = b - 1$ yields the desired formula.

The case $t = 5$: One has, by the case $t = 3$,

$$\partial(\delta_{2a-3}\delta_a) = \partial(\delta_{2a-2}\delta_{a-1})$$

which yields

$$\delta_{2a-3}\delta_{a+1} = \delta_{2a-1}\delta_{a-1}.$$

Again setting $a = b - 1$ yields the desired result.

The case $t = 6$ is similar.

By now the pattern is clear. Fix $t \geq 7$. Apply ∂ to

$$\delta_{2b-t+2}\delta_b = \sum_{(t-2)/3 \leq s \leq (t-3)/2} \binom{t-s-3}{s} \delta_{2b-t+2+s}\delta_{b-s}$$

and use the fact that

$$\delta_{2b-t+3}\delta_b = \sum_{(t-3)/3 \leq s \leq (t-4)/2} \binom{t-s-4}{s} \delta_{2b-t+3+s} \delta_{b-s}$$

to obtain a formula

$$\begin{aligned} \delta_{2b-t+2}\delta_{b+1} &= \sum_{(t-3)/3 \leq s \leq (t-4)/2} \binom{t-s-4}{s} \delta_{2b-t+3+s} \delta_{b-s} \\ &+ \sum_{(t-2)/3 \leq s \leq (t-3)/2} \binom{t-s-3}{s} \delta_{2b-t+3+s} \delta_{b-s} \\ &+ \sum_{(t-5)/3 \leq s \leq (t-5)/2} \binom{t-s-4}{s+1} \delta_{2b-t+3+s} \delta_{b-s}. \end{aligned}$$

The difficulty comes in simplifying this formula, mostly because of the end-points of the three summations are all different. We break this into six cases, one for each congruence class of t modulo 6. We do one case; the other cases are similar. So assume that $t = 6a + 2$ and $a \geq 1$. Then

$$\begin{aligned} \delta_{2b-t+2}\delta_{b+1} &= \sum_{2a \leq s \leq 3a-1} \binom{t-s-4}{s} \delta_{2b-t+3+s} \delta_{b-s} \\ &+ \sum_{2a \leq s \leq 3a-1} \binom{t-s-3}{s} \delta_{2b-t+3+s} \delta_{b-s} \\ &+ \sum_{2a-1 \leq s \leq 3a-2} \binom{t-s-4}{s+1} \delta_{2b-t+3+s} \delta_{b-s}. \end{aligned}$$

This can be rewritten as

$$\begin{aligned} \delta_{2b-t+2}\delta_{b+1} &= \binom{t-2a-3}{2a} \delta_{2b-4a} \delta_{b-2a+1} \\ &+ \sum_{2a \leq s \leq 3a-2} \binom{t-s-2}{s+1} \delta_{2b-t+s+3} \delta_{b-s} \\ &+ \binom{3a-1}{3a-2} \delta_{2b-3a} \delta_{b-3a+1}. \end{aligned}$$

Since, by the case $t = 2$, $\delta_{2b-4a} \delta_{b-2a+1} = 0$ and $\binom{3a+1}{3a} = \binom{3a-1}{3a-2}$, we have

$$\delta_{2b-t+2}\delta_{b+1} = \sum_{2a \leq s \leq 3a-1} \binom{t-s-2}{s+1} \delta_{2b-t+s+3} \delta_{b-s}$$

or

$$\delta_{2b-t}\delta_b = \sum_{2a+1 \leq s \leq 3a} \binom{t-s-1}{s} \delta_{2b-t+s} \delta_{b-s}$$

as predicted.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195
E-mail address: pgoerss@math.washington.edu

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695
E-mail address: lada@math.ncsu.edu