PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

# CONFORMAL DIFFEOMORPHISMS PRESERVING THE RICCI TENSOR

W. KÜHNEL AND H.-B. RADEMACHER

(Communicated by Christopher B. Croke)

ABSTRACT. We characterize semi-Riemannian manifolds admitting a global conformal transformation such that the difference of the two Ricci tensors is a constant multiple of the metric. Unless the conformal transformation is homothetic, the only possibilities are standard Riemannian spaces of constant sectional curvature and a particular warped product with a Ricci flat Riemannian manifold.

We consider semi-Riemannian manifolds $(M, g)$ and *conformal diffeomorphisms* $f: (M, g) \to (\overline{M}, \overline{g})$ between them meaning that $f^*\overline{g}$ is pointwise a positive scalar multiple of $g$. If this factor is constant $f$ is called a *homothety.* A *conformal structure* on $M$ is a class of conformally equivalent metrics. In this paper we study the behavior of the Ricci tensor $\mathrm{Ric}_g$ within one conformal class A classical theorem of Liouville determines all possible conformal diffeomorphisms between euclidean metrics. As a generalization we call a conformal transformation $g \to \overline{g}$ a *Liouville transformation* if $\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = 0$. In Theorems 1 and 2 we classify complete semi-Riemannian manifolds $(M, g)$ admitting a non-homothetic conformal transformation $\overline{g} = \varphi^{-2}g$ such that the difference $\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g$ of the Ricci tensors is a constant multiple of the metric $g$ or $\overline{g}$. We show that $M$ is Riemannian and that $M$ is either a standard space of constant sectional curvature or is a warped product $\mathbb{R} \times_{\exp} M_*$ of the real line and a Ricci flat manifold $M_*$. As a special case we obtain in Corollary 1 that a globally defined Liouville transformation is a homothety. This result is due to Liouville [Liv] in the case of $E^3$, generalized by Lie [Lie] to the case of $E^n$ (see also [S, p. 173]) and by Haantjes [H] to the case of pseudo-euclidean space $E^n_k$. For Riemannian manifolds Corollary 1 has been obtained by Ferrand [Fe], where a Liouville transformation is called a quasi-similarity. For the compact Riemannian case, compare the recent paper [X] but be careful with the signs in formula (2) there. In Theorem 3 we show that for a complete semi-Riemannian manifold admitting a global non-homothetic concircular transformation between two metrics of constant scalar curvature the same conclusions as in Theorem 1 hold.

Received by the editors August 25, 1993 and, in revised form, February 3, 1994.
1991 *Mathematics Subject Classification.* Primary 53C20, 53C50; Secondary 53A30, 58G30.
*Key words and phrases.* Semi-Riemannian manifolds, Ricci tensor, conformal mapping, Hessian.

*General assumption.* $(M^n, g)$ is a semi-Riemannian manifold of dimension $n \geq 3$ and of class $C^3$ such that the number of negative eigenvalues of $g$ is not greater than $\frac{n}{2}$.

*Notation.* $\operatorname{grad}\varphi$ denotes the gradient of $\varphi$, $\nabla^2\varphi = \nabla \operatorname{grad}\varphi$ is the Hessian of $\varphi$, and $\Delta\varphi = \operatorname{trace}\nabla^2\varphi$ is the Laplacian. Let us introduce the notation $[h]$ for the class of all tensors which are pointwise scalar multiples of a given $(0, 2)$-tensor $h$. This includes the zero tensor—as well as negative multiples—which is not in the conformal structure itself. $(M, g)$ is called an *Einstein space* if $[\mathrm{Ric}_g] = [g]$. Two metrics $g, \overline{g}$ are *concircular* to one another if $[g] = [\overline{g}] = [\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g]$.

**Lemma 1.** *Two conformally equivalent metrics $g$ and $\overline{g} = \frac{1}{\varphi^2}g$ satisfy the relation*

$$[\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g] = [g] = [\overline{g}]$$

*if and only if the function $\varphi$ satisfies the equation*

$$\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g.$$

*Proof.* This follows from the well-known formula [S, p. 168], [Be, Sect. 1J], [Kü, Sect. A]

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \frac{1}{\varphi^2}[(n-2)\varphi\nabla^2\varphi + (\varphi\Delta\varphi - (n-1)g(\operatorname{grad}\varphi, \operatorname{grad}\varphi)) \cdot g].$$

Lemma 1 holds only under the assumption $n \geq 3$.

*Remark.* A transformation $g \to \overline{g}$ as in Lemma 1 is called *concircular* because it preserves the curves of constant geodesic curvature and vanishing geodesic torsion (so-called *geodesic circles*); see [Y]. In this case the conformal geodesics are geodesic circles; see [Fi, p. 454]. Any conformal transformation between two Einstein spaces is automatically concircular. A concircular transformation $g \to \overline{g}$ satisfies $\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \frac{1}{n}(\frac{\overline{S}}{\varphi^2} - S) \cdot g$ where $S, \overline{S}$ denote the scalar curvatures of $g, \overline{g}$.

**Lemma 2.** *A function $\varphi\colon M \to \mathbb{R}$ satisfies $\nabla^2\varphi = \lambda \cdot g$ for some $\lambda\colon M \to \mathbb{R}$ in a neighborhood of a point with $g(\operatorname{grad}\varphi, \operatorname{grad}\varphi) \neq 0$ if and only if $g$ is locally a warped product metric $ds^2 = \eta dt^2 + \varphi^2(t)ds_*^2$, where $\eta \in \{+1, -1\}$ denotes the sign of $g(\operatorname{grad}\varphi, \operatorname{grad}\varphi)$, $\varphi, \lambda$ are functions depending only on $t$ satisfying $\varphi'' = \eta \cdot \lambda$, and $ds_*^2$ is independent of $t$.*

This lemma can be found in [Fi, Sect. 12] and also in [T] for the Riemannian case. For the special case of Einstein metrics it is due to Brinkmann [Br].

Particular consequences of Lemma 2 are the following:

1. If $\nabla^2\varphi = \lambda g$ and if $\operatorname{grad}\varphi$ is a non-null vector, then the trajectories of $\operatorname{grad}\varphi$ are geodesics (up to parametrization).

2. If $\nabla^2\varphi = \lambda g$, then along every such non-null geodesic $\gamma(t)$ in direction $\operatorname{grad}\varphi$ with $g(\dot\gamma, \dot\gamma) = \eta$ the function $\varphi(t) := \varphi(\gamma(t))$ satisfies $\varphi'' = \eta \cdot \lambda$. Along a null geodesic this function $\varphi(t)$ satisfies $\varphi'' = 0$ according to the proof of Lemma 3 below.

**Definition.** A semi-Riemannian manifold $(M, g)$ is called (geodesically) *complete* if every geodesic can be defined over $\mathbb{R}$ [O, p. 68]. It is called *null complete* if this holds for every null geodesic.

**Theorem 1.** *Let $(M, g)$ be complete and admitting a global conformal transformation $\overline{g} = \frac{1}{\varphi^2} \cdot g$ satisfying*

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = c \cdot (n-1) \cdot g$$

*for some constant $c$. Then one of the following three cases occurs:*

1. *$\varphi$ is constant.*
2. *$(M, g)$ and $(M, \overline{g})$ are simply connected Riemannian spaces of constant sectional curvature.*
3. *$(M, g)$ is a warped product $\mathbb{R} \times_{e^t} M_*$ where $\varphi(t) = e^t$ and $(M_*, g_*)$ is a complete Ricci-flat $(n-1)$-dimensional Riemannian manifold.*

*Remark.* In Theorem 1 case 1 corresponds to $c = 0$, and cases 2 and 3 correspond to $c > 0$. In case 2 $\overline{g}$ must be flat and $g$ must be hyperbolic, and in case 3 $\mathrm{Ric}_{\overline{g}} = 0$. A non-constant $\varphi$ occurs only for Einstein space. Therefore we compare this to another theorem.

**Theorem 1*.** *Let $(M, g)$ be complete and assume that both $g$ and $\overline{g} = \frac{1}{\varphi^2} \cdot g$ are Einstein metrics. Then the same conclusion as in Theorem 1 holds, i.e. one of the cases* 1, 2, 3 *occurs.*

*Remark.* In Theorem 1* case 2 occurs for various combinations of the signs of the constant curvatures of $g$ and $\overline{g}$.

**Theorem 2.** *Let $(M, g)$ be complete and admitting a global conformal transformation $\overline{g} = \frac{1}{\varphi^2} \cdot g$ satisfying*

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = c \cdot (n-1) \cdot \overline{g} = \frac{c \cdot (n-1)}{\varphi^2} \cdot g$$

*for a constant $c$. Then either $\varphi$ is constant or $(M, g)$ is isometric with the euclidean space.*

*Remark.* To pass from Theorem 1 to Theorem 2 one just has to interchange the roles of $g$ and $\overline{g}$. However, this is not quite symmetric because at most one of them can be complete (unless $\varphi$ is constant). In Theorem 2 a constant $\varphi$ corresponds to $c = 0$; the other case corresponds to $c > 0$. In particular, $c < 0$ is impossible in Theorem 1 and in Theorem 2.

**Corollary 1.** *A globally defined Liouville transformation of a complete semi-Riemannian manifold is a homothety.*

This result is just the case $c = 0$ in Theorems 1 and 2.

**Corollary 2.** *Assume that two semi-Riemannian metrics in the same conformal class have pointwise the same Ricci tensor. If one of them is complete, then they are homothetic to one another.*

This is a uniqueness result for the problem of prescribing a Ricci tensor in a conformal class.

**Lemma 3.** *Let $g$ be a null complete indefinite metric admitting a globally defined nonconstant solution $\varphi$ of $\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g$. Then $\varphi$ has a zero.*

*Proof*. Along any null geodesic $\gamma(s)$ one calculates

$$\begin{aligned}\frac{d^2}{ds^2}(\varphi(\gamma(s))) &= \frac{d}{ds}g(\operatorname{grad}\varphi, \dot\gamma)\\ &= g(\nabla_{\dot\gamma}\operatorname{grad}\varphi, \dot\gamma) + g(\operatorname{grad}\varphi, \nabla_{\dot\gamma}\dot\gamma)\\ &= g\left(\frac{\Delta\varphi}{n}\cdot\dot\gamma, \dot\gamma\right)\\ &= 0.\end{aligned}$$

Therefore $\varphi(\gamma(s))$ is linear in $s$ with $\frac{d}{ds}\varphi(\gamma(s)) = g(\operatorname{gradf}\varphi, \dot\gamma)$. If we choose $\gamma$ such that $g(\operatorname{grad}\varphi, \dot\gamma) \neq 0$ at a point $p$, then it follows that $\varphi$ has a zero along $\gamma$.

**Corollary 3.** *The only globally defined concircular transformations of a null complete indefinite metric are the homotheties.*

**Theorem 3.** *Let $(M, g)$ be complete and admitting a globally defined concircular transformation $\overline{g} = \frac{1}{\varphi^2} \cdot g$. Assume that $S, \overline{S}$ are constant. Then one of the three cases* 1, 2, 3 *as in Theorem* 1 *occurs.*

*Proof of Theorem* 1. By assumption $\varphi\colon M \to \mathbb{R}$ is a function which is positive everywhere. By Lemma 3 we may assume that $\varphi$ is a non-constant function on a manifold with positive definite metric $g$. Let $p \in M$ be a point with $\operatorname{grad}\varphi(p) \neq 0$. By Lemma 1 the equation

$$\begin{aligned}c(n-1)\cdot g &= \mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g\\ &= \frac{1}{\varphi^2}[(n-2)\varphi\cdot\nabla^2\varphi + (\varphi\Delta\varphi - (n-1)\|\operatorname{grad}\varphi\|^2)\cdot g]\end{aligned}$$

implies

$$\frac{2(n-1)}{n}\varphi\Delta\varphi - (n-1)\|\operatorname{grad}\varphi\|^2 - c(n-1)\varphi^2 = 0. \tag{1}$$

By Lemma 2 along the unit speed geodesic $\gamma(t)$ in the direction of $\operatorname{grad}\varphi = \varphi' \cdot \frac{\partial}{\partial t}$ we have

$$2\varphi\varphi'' - \varphi'^2 - c\varphi^2 = 0. \tag{2}$$

Differentiating once more we get

$$2\varphi\varphi''' - 2c\varphi\varphi' = 0 \tag{3}$$

or, since $\varphi \neq 0$,

$$\varphi''' = c \cdot \varphi'. \tag{4}$$

Therefore there is a constant $a$ satisfying

$$\varphi'' = c\cdot\varphi + a. \tag{5}$$

$\varphi'(p) \neq 0$ implies equivalently

$$(\varphi'^2)' = c\cdot(\varphi^2)' + 2a\varphi'. \tag{6}$$

This means that there is a constant $b$ satisfying

$$\varphi'^2 = c \cdot \varphi^2 + 2a\varphi + b. \tag{7}$$

Now (5), (7), and (2) together imply $b = 0$; thus

$$\varphi'^2 = \varphi(c \cdot \varphi + 2a). \tag{8}$$

*Case* I: $c = 0$. By (5) $\varphi$ is a polynomial of degree at most 2. Then (8) reduces to $\varphi'^2 = 2a\varphi$, which implies that $\varphi$ is quadratic and that $\varphi$ and $\varphi'$ have a common zero along $\gamma$. By the completeness of $g$ the metric $\overline{g} = \frac{1}{\varphi^2} \cdot g$ has a singularity there, a contradiction. Alternatively, if $\varphi(t) = At^2 + Bt + C$, then (8) implies that the discriminant $4AC - B^2$ is zero. Therefore $\varphi(t)$ is the square of a linear function.

*Case* II: $c < 0$. In this case every solution $\varphi$ of (5) is periodic and therefore attains its minimum and maximum. At each of these points the equation $0 = \varphi(c\varphi + 2a)$ is satisfied by (8). Hence $\varphi = 0$ and $c\varphi + 2a = 0$ must be satisfied at the minimum and maximum, respectively. This leads to a contradiction as in Case I.

Alternatively, the general solution of (5)

$$\varphi(t) = \alpha \cos(\sqrt{-c}t) + \beta \sin(\sqrt{-c}t) - \frac{a}{c}$$

satisfies

$$\alpha^2 + \beta^2 = \frac{a^2}{c^2}$$

by (8). A typical solution looks like $\varphi(t) = \cos t + 1$.

*Case* III: $c > 0$. In this case the general solution of (5) is

$$\varphi(t) = \alpha\cosh(\sqrt{c}t) + \beta\sinh(\sqrt{c}t) - \frac{a}{c}.$$

Then (8) implies

$$\alpha^2 - \beta^2 = \frac{a^2}{c^2},$$

in particular $\alpha^2 \geq \beta^2$.

If $\alpha^2 > \beta^2$, then $\varphi$ has a critical point along $\gamma$. This implies that $\gamma$ has a point $q$ with $\operatorname{grad}\varphi(q) = 0$ (note that $\operatorname{grad}\varphi = \varphi' \cdot \frac{\partial}{\partial t}$). Furthermore $\varphi$ satisfies globally $\nabla^2\varphi = (c\varphi + a) \cdot g$ with $c > 0$. A result of Tashiro [T] implies that $(M, g)$ is isometric with the hyperbolic space of constant sectional curvature $-c$. Roughly the argument is the following: In geodesic polar coordinates around $q$ the metric $g$ looks like $ds^2 = dt^2 + \sinh^2(\sqrt{c}t) \cdot ds_1^2$ where $ds_1^2$ is the metric of a round sphere of appropriate radius; compare [Kü, Lemma 18].

If $\alpha^2 = \beta^2$, then $a = 0$ and $\varphi(t) = \alpha \cdot e^{\mp\sqrt{c}\cdot t}$ is a solution without a critical point along $\gamma$. This implies that

$$ds^2 = dt^2 + e^{2\sqrt{c}\cdot t} ds_*^2 \tag{9}$$

is a complete metric on $M = \mathbb{R} \times M_*$. It follows that $\overline{g} = e^{-2\sqrt{c}t} g$ is the product metric $dt^2 + ds_*^2$ on $(0, \infty) \times M_*$. For an arbitrary tangent vector

$X$ in the direction of $M_*$ the standard formulas for the curvature of warped products imply

$$\begin{aligned} c(n-1)g(X, X) &= (\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g)(X, X) \\ &= (1 - c \cdot e^{2\sqrt{c}t})\mathrm{Ric}_{g_*}(X, X) + c(n-1)g(X, X), \end{aligned}$$

which is impossible unless $\mathrm{Ric}_{g_*} = 0$. This completes the proof of Theorem 1.

Note that in the case of an indefinite metric (9) does not define a complete warped product metric; compare [O, p. 209]. Compare also [Kb] for global solutions of $\nabla^2\varphi = c \cdot \varphi \cdot g$, $c > 0$, in the indefinite case if $\varphi$ has at least one critical point.

*Proof of Theorem* 1*. In the case of indefinite metrics $\varphi$ is constant by Lemma 3, using the same argument as in Theorem 1. The Riemannian case has been treated in [Kü, Theorem 27]. The local considerations in this case are due to Brinkmann [Br]. Compare also [Be, 9.110].

*Remark.* Geodesic mappings of the same kind as in Theorem 1 have been studied in [V]. For the case of conformal vector fields on Einstein spaces compare [YN] and [Kan] in the Riemannian case and [Ke1], [Ke2] in the non-Riemannian case. Brinkmann describes in [Br, §4] indefinite Einstein metrics $(M, g)$ carrying a non-constant positive function $\varphi$ such that the conformally equivalent metric $\overline{g} := \varphi^{-2}g$ is also Einstein and the gradient $\operatorname{grad}\varphi$ is everywhere null. Then it follows that $\mathrm{Ric}_{\overline{g}} = \mathrm{Ric}_g = 0$ and $\nabla^2\varphi = 0$, i.e. $\operatorname{grad}\varphi$ is parallel. By Lemma 3 $(M, g)$ cannot be null complete.

In general relativity these metrics were studied in several papers. They are called *pp-waves* or *gravitational plane waves*; see e.g. [Hf].

Locally all the considerations in the proofs of Theorems 1–3 remain valid also in the case of an indefinite metric. This includes a local classification and the existence of various examples which, however, cannot be null complete.

*Proof of Theorem* 2. This follows the pattern of the proof of Theorem 1. In particular $g$ must be positive definite if $\varphi$ is not constant. We start with the equation

$$\frac{c \cdot (n-1)}{\varphi^2} \cdot g = \mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \frac{1}{\varphi^2}[(n-2)\varphi \cdot \nabla^2\varphi + (\varphi\Delta\varphi - (n-1)\|\operatorname{grad}\varphi\|^2) \cdot g]$$

which implies

$$(10) \qquad \frac{2(n-1)}{n}\varphi\Delta\varphi - (n-1)\|\operatorname{grad}\varphi\|^2 - c(n-1) = 0.$$

If $\operatorname{grad}\varphi \neq 0$ at $p$, then along the geodesic $\gamma$ in direction $\operatorname{grad}\varphi$ we have

$$(11) \qquad 2\varphi\varphi'' - \varphi'^2 - c = 0,$$

which implies

$$(12) \qquad 2\varphi\varphi''' = 0$$

or

$$(13) \qquad \varphi(t) = At^2 + Bt + C.$$

If we put this into (11) we get

$$(14) \qquad 4AC - B^2 = c.$$

The case $c = 0$ leads to a zero of $\varphi$ as in the proof of Theorem 1; the case $c < 0$ leads to two zeros of $\varphi$, a contradiction. If $c > 0$, then $\varphi$ has no zero but it has a critical point along $\gamma$. This is a critical point for $\varphi$ on $M$. $\varphi$ satifies the equation $\nabla^2\varphi = 2A \cdot g$. By a theorem of Tashiro [T] this implies that $(M, g)$ is isometric with the euclidean space. Around the critical point the geodesic polar coordinates coincide with the euclidean polar coordinates.

In particular, if $\varphi$ is non-constant, then $c$ must be positive and $\overline{g}$ is a space of constant sectional curvature $c$.

*Proof of Theorem* 3. The case of an indefinite metric can be ruled out by Lemma 3. In the Riemannian case the equation

$$\mathrm{Ric}_{\overline{g}} - \mathrm{Ric}_g = \frac{1}{n}\left(\frac{\overline{S}}{\varphi^2} - S\right) \cdot g \tag{15}$$

implies

$$2\varphi\varphi'' - \varphi'^2 + \frac{S}{n(n-1)} \cdot \varphi^2 - \frac{\overline{S}}{n(n-1)} = 0 \tag{16}$$

along a unit speed geodesic in direction $\operatorname{grad}\varphi$. Differentiating once more leads to

$$2\varphi\varphi''' + \frac{2S}{n(n-1)}\varphi\varphi' = 0$$

or

$$\varphi''' + \rho \cdot \varphi' = 0 \tag{17}$$

where $\rho := \frac{S}{n(n-1)}$ denotes the normalized scalar curvature.

As in the proof of Theorem 1 we conclude

$$\varphi'^2 = -\rho\varphi^2 + 2a \cdot \varphi - \overline{\rho} \tag{18}$$

for a certain constant $a$.

In any case the solution $\varphi$ of (17) and (18) either has a zero (which is impossible because $\varphi$ is a conformal factor) or a critical point, except for solutions of the type

$$\varphi'(t) = \alpha \cdot e^{\sqrt{-\rho}t} \tag{19}$$

leading to the same warped product metric as in (9). If there is a critical point, then the levels around it are round spheres and thus $(M, g)$ is a standard space of constant sectional curvature [T], [Kü, Lemmas 13 and 18]. This completes the proof of Theorem 3.

The local part of this calculation is due to Tachibana [Tb, Theorem 8.1]. In the compact case the following holds: a compact Riemannian manifold with constant scalar curvature admitting a non-constant solution of

$$\nabla^2\varphi = \frac{\Delta\varphi}{n} \cdot g$$

is isometric with a round sphere [Kü, Theorem 24].

## References


[Be] A. L. Besse, *Einstein manifolds*, Ergebnisse Math. Grenzgeb., 3. Folge, Band 10, Springer, Berlin, Heidelberg, and New York, 1987.

[Br] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145.

[Fe] J. Ferrand, *Sur une classe de morphismes conformes*, C. R. Acad. Sci. Paris **296** (1983), 681–684.

[Fi] A. Failkow, *Conformal geodesics*, Trans. Amer. Math. Soc. **45** (1939), 443–473.

[H] J. Haantjes, *Conformal representations of an n-dimensional euclidean space with a non-dimensional euclidean space with a non-definite fundamental form on itself*, Proc. Kon. Nederl. Akad. Amsterdam **40** (1937), 700–705.

[Hf] W. D. Halford, *Brinkmann's theorem in general relativity*, Gen. Relativity Gravitation **14** (1982), 1193–1195.

[Kan] M. Kanai, *On a differential equation characterizing a Riemannian structure of a manifold*, Tokyo J. Math. **6** (1983), 143–151.

[Kb] Y. Kerbrat, *Transformations conformes des variétés pseudo-Riemanniennes*, J. Differential Geom. **11** (1976), 547–571.

[Ke1] M. G. Kerckhove, *Conformal transformations of pseudo-Riemannian Einstein manifolds*, thesis, Brown Univ., 1988.

[Ke2] ——, *The structure of Einstein spaces admitting conformal motions*, Classical Quantum Gravity **8** (1991), 819–825.

[Kü] W. Kühnel, *Conformal transformations between Einstein spaces*, Conformal Geometry (R. S. Kulkarni and U. Pinkall, eds.), Aspects of Math., vol. E12, Braunschweig, 105–146, Vieweg-Verlag, 1988, pp. 105–146.

[Lie] S. Lie, *Komplexe, insbesondere Linien und Kugelkomplexe mit Anwendung auf die Theorie partieller Differentialgleichungen*, Math. Ann. **5** (1872), 145–246.

[Liv] J. Liouville, *Extension au cas des trois dimensions de la question du tracé géographique*, Note VI, Applications de l'Analyse à la Géométrie (G. Monge, ed.), Paris, 1850, pp. 609–617.

[O] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[S] J. A. Schouten, *Der Ricci-Kalkül*, Springer-Verlag, Berlin, 1924.

[Tb] S. Tachibana, *On concircular geometry and Riemann spaces with constant scalar curvatures*, Tôhoku Math. J. **3** (1951), 149–158.

[T] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. **117** (1965), 251–275.

[V] P. Venzi, *Klassifikation der geodätischen Abbildungen mit* $\overline{\mathrm{Ric}} - \mathrm{Ric} = \Delta \cdot g$, Tensor (N.S.) **37** (1982), 137–147.

[X] X. Xu, *Prescribing a Ricci tensor in a conformal class of Riemannian metrics*, Proc. Amer. Math. Soc. **115** (1992), 455–459; corrigenda, ibid. **118** (1993), 333.

[Y] K. Yano, *Concircular geometry* I–V, Proc. Imperial Acad. Japan **16** (1940), 195–200, 354–360, 442–448, 505–511; ibid. **18** (1942), 446–451.

[YN] K. Yano and T. Nagano, *Einstein spaces admitting a one-parameter group of conformal transformations*, Ann. of Math. (2) **69** (1959), 451–460.



Fachbereich Mathematik, Universität Duisburg, 47048-Duisburg, Germany
*Current address*: Mathematisches Institut B, Universität Stuttgart, 70550 Stuttgart, Germany
*E-mail address*: `kuehnel@morse.physik.uni-stuttgart.de`

Institut für Mathematik, Universität Augsburg, 86135-Augsburg, Germany
*Current address*: Mathematisches Institut, Universität Leipzig, 04009 Leipzig, Germany
*E-mail address*: `rademacher@uni-augsburg.de`