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COMPLETE HYPERSURFACES WITH CONSTANT MEAN CURVATURE AND NON-NEGATIVE SECTIONAL CURVATURES

HU ZE-JUN

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ABSTRACT. We classify the complete and non-negatively curved hypersurfaces of constant mean curvature in spaces of constant sectional curvature.

1. INTRODUCTION

Let $\overline{M}^{n+1}(c)$ be an $(n+1)$-dimensional space of constant sectional curvature c. When $c<0$, $\overline{M}^{n+1}(c)=H^{n+1}(c)$; when $c=0$, $\overline{M}^{n+1}(c)=R^{n+1}$; when $c>0$, $\overline{M}^{n+1}(c)=S^{n+1}(c)$, respectively. Let M^n be an n-dimensional hypersurface with constant mean curvature H in $\overline{M}^{n+1}(c)$. Let S denote the square of the length of the second fundamental form. The main purpose of this paper is to give a characterization of non-negatively curved hypersurfaces of $\overline{M}^{n+1}(c)$ by the relationship between S and H, this can be compared with the results obtained by Nomizu and Smyth [3], Okumura [4], Goldberg [1], Hasanis [2] and Smyth [6].

Theorem. *Let M^n be a complete non-negatively curved hypersurface of $\overline{M}^{n+1}(c)$ with constant mean curvature H. Then M^n is totally umbilical or*

$$\sup S = nc+\frac{n^3H^2}{2k(n-k)}\pm\frac{n(n-2k)}{2k(n-k)}|H|\sqrt{n^2H^2+4k(n-k)c},$$

for some $k=1,2,\dots,n-1$, when $c\geq 0$; or

$$\sup S = nc+\frac{n^3H^2}{2(n-1)}-\frac{n(n-2)}{2(n-1)}|H|\sqrt{n^2H^2+4(n-1)c},\quad \text{when } c<0.$$

In particular, if M^n is connected and $S=$ constant (when $c>0$, $S=$ constant may be replaced by the condition that M^n is compact), then in the second case we have the following:

(1) *When $c>0$, $M^n=S^k(c_1)\times S^{n-k}(c_2)$, for some $k=1,2,\dots,n-1$, where $c_1>0$, $c_2>0$ and $1/c_1+1/c_2=1/c$.*

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(2) *When* $c = 0$, $M^n = R^k \times S^{n-k}(c_1)$ *for some* $k = 1, 2, \dots, n-1$, *where* $c_1 > 0$.
(3) *When* $c < 0$, $M^n = H^1(c_1) \times S^{n-1}(c_2)$, *where* $c_1 < 0$, $c_2 > 0$ *and* $1/c_1 + 1/c_2 = 1/c$.

From the theorem, we easily see the following

Corollary. *Let* M^n *be a compact non-negatively curved hypersurface of* $\overline{M}^{n+1}(c)$ $(c \leq 0)$ *with constant mean curvature* H. *Then* M^n *is totally umbilical and has positive sectional curvature* $H^2 + c$.

Remark. The main theorem was partially proved by Nomizu and Smyth in [3] whose results were extended to arbitrary codimension by Smyth [6] and later by Yau in [7] under the condition that M^n is compact.

2. Preliminaries

Let M^n be a hypersurface of $\overline{M}^{n+1}(c)$ and let $e_1, \dots, e_n, e_{n+1}$ be a local field of orthonormal frames in $\overline{M}^{n+1}(c)$, such that, restricted to M^n, the vector field e_{n+1} is normal to M^n. Then the second fundamental form B and the mean curvature H for M^n can be written as

$$B = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}, \qquad H = (1/n)\sum_i h_{ii}. \tag{2.1}$$

The Gauss equation for M^n is

$$R = n(n-1)c + n^2H^2 - S, \tag{2.2}$$

where $S = \operatorname{tr} B^2 = \sum_{i,j} h_{ij}^2$ and R denotes the scalar curvature of M^n.

We denote by Δ the Laplacian relative to the induced metric on M^n. If $H =$ constant, then ([3])

$$(1/2)\Delta S = |\nabla B|^2 - S^2 + ncS - n^2cH^2 + nH\operatorname{tr} B^3. \tag{2.3}$$

For any point p in M^n, we can choose a local frame field $e_1, e_2, \dots, e_n$ so that the matrix (h_{ij}) is diagonalized at that point, say, $h_{ij} = \lambda_i\delta_{ij}$. Then (2.3) can be rewritten as ([3])

$$(1/2)\Delta S = |\nabla B|^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2 K_{ij}, \tag{2.4}$$

where $K_{ij} = c + \lambda_i\lambda_j$ is the sectional curvature of the plane section spanned by e_i and e_j.

Lemma (see Omori [5] and Yau [8]). *Let* M^n *be an* n*-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let* F *be a* C^2*-function bounded from above on* M^n. *Then for any* $\varepsilon > 0$, *there exists a point* p *in* M^n *such that*

$$\sup F - \varepsilon < F(p), \quad |\operatorname{grad} F|(p) < \varepsilon, \quad \Delta F(p) < \varepsilon.$$

3. Standard models

This section is concerned with some standard models of complete non-negatively curved hypersurfaces with constant mean curvature of the space form

$\overline{M}^{n+1}(c)$. In particular, we only consider non-totally umbilical cases, and the length of the second fundamental form of such hypersurfaces are calculated.

First, we consider a class of hypersurfaces $R^k \times S^{n-k}(c_1)$ of R^{n+1}, where $k = 1, 2, \dots, n-1$. The number of distinct principal curvatures of each hypersurface is exactly two, say 0 and $\sqrt{c_1}$, with multiplicities k and $n-k$, respectively. The sectional curvatures of a plane spanned by two principal directions are 0 and c_1, respectively. It is easily seen that H and S are constant, and they satisfy, for $R^k \times S^{n-k}(c_1)$ in R^{n+1}, $S = n^2H^2/(n-k)$.

We next consider the case $c > 0$. Let

$$\begin{aligned} S^k(c_1) &= \{(x_1, \dots, x_{k+1}) \in R^{k+1};\ x_1^2 + \cdots + x_{k+1}^2 = 1/c_1\}, \\ S^{n-k}(c_2) &= \{(y_1, \dots, y_{n-k+1}) \in R^{n-k+1};\ y_1^2 + \cdots + y_{n-k+1}^2 = 1/c_2\}, \\ S^{n+1}(c) &= \{(x_1, \dots, x_{k+1}, y_1, \dots, y_{n-k+1}) \in R^{n+2}; \\ &\qquad x_1^2 + \cdots + x_{k+1}^2 + y_1^2 + \cdots + y_{n-k+1}^2 = 1/c\}, \end{aligned}$$

where $1/c_1 + 1/c_2 = 1/c$, $k = 1, 2, \dots, n-1$. Then $S^k(c_1) \times S^{n-k}(c_2)$ is a family of hypersurfaces in $S^{n+1}(c)$. The number of distinct principal curvatures of such hypersurfaces in this family is exactly two, and they are constant. One principal curvature is equal to $\pm\sqrt{c_1 - c}$ with multiplicity k, and the other is equal to $\mp\sqrt{c_2 - c}$ with multiplicity $n-k$. The sectional curvatures of a plane spanned by two principal directions are c_1, c_2 and 0, respectively. For $S^k(c_1) \times S^{n-k}(c_2)$ in $S^{n+1}(c)$ we can easily show that

$$S = nc + \frac{n^3H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)}|H|\sqrt{n^2H^2 + 4k(n-k)c},$$

where the plus (resp. minus) sign is taken if $k\sqrt{c_1 - c} \geq (n-k)\sqrt{c_2 - c}$ (resp. $k\sqrt{c_1 - c} < (n-k)\sqrt{c_2 - c}$).

Finally, we consider the case $c < 0$. Let

$$H^{n+1}(c) = \{(x_0, x_1, \dots, x_{n+1}) \in R_1^{n+2};\ -x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = 1/c\}.$$

Define a family of hypersurfaces $H^k(c_1) \times S^{n-k}(c_2)$ in $H^{n+1}(c)$ by

$$\begin{aligned} H^k(c_1) \times S^{n-k}(c_2) = \{&(x_0, x_1, \dots, x_{n+1}) \in R_1^{n+2}; \\ &-x_0^2 + x_1^2 + \cdots + x_k^2 = 1/c_1,\ x_{k+1}^2 + \cdots + x_{n+1}^2 = 1/c_2\}, \end{aligned}$$

where $c_1 < 0$, $c_2 > 0$, $1/c_1 + 1/c_2 = 1/c$ and $k = 1, 2, \dots, n-1$. The number of distinct principal curvatures of such hypersurfaces in this family is exactly two and they are constant. One principal curvature is equal to $\sqrt{c_1 - c}$ with multiplicity k and the other is equal to $\sqrt{c_2 - c}$ with multiplicity $n-k$. The sectional curvatures are $0, c_2$ and c_1 for $1 < k < n$; they are 0 and c_2 for $k = 1$. For $H^k(c_1) \times S^{n-k}(c_2)$ in $H^{n+1}(c)$, we can also easily show that

$$S = nc + \frac{n^3H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)}|H|\sqrt{n^2H^2 + 4k(n-k)c},$$

where the plus (resp. minus) sign is taken if $k\sqrt{c_1 - c} \geq (n-k)\sqrt{c_2 - c}$ (resp. $k\sqrt{c_1 - c} < (n-k)\sqrt{c_2 - c}$).

4. Proof of Theorem

Since $K_{ij} = c + \lambda_i\lambda_j \geq 0$ for any distinct indices i and j, (2.4) means that

$$(1/2)\Delta S = |\nabla B|^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j) \geq 0. \tag{4.1}$$

On the other hand, $K_{ij} \geq 0$ implies that the Ricci curvature of M^n is bounded from below by zero. From (2.2) we have that S is bounded from above by a constant $n(n-1)c + n^2H^2$. Then we can apply the lemma to the function S. Then we get $\{p_m\}$ in M^n such that

$$\lim_{m\to\infty} S(p_m) = \sup S, \quad \lim_{m\to\infty} \Delta S(p_m) \leq 0. \tag{4.2}$$

This implies that (4.1) and (4.2) give rise to

$$(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2(p_m) \to 0 \qquad (m \to \infty), \tag{4.3}$$

for any distinct indices i and j.

Now, since $S = \sum \lambda_j^2$ is bounded, any principal curvature λ_j is bounded and hence so is any sequence $\{\lambda_j(p_m)\}$. Then there exists a subsequence $\{p_{m'}\}$ of $\{p_m\}$ such that

$$\lambda_j(p_{m'}) \to \mathring{\lambda}_j \qquad (m' \to \infty) \text{ for some } \mathring{\lambda}_j \text{ and any } j. \tag{4.4}$$

In fact, since a sequence $\{\lambda_1(p_m)\}$ is bounded, it converges to some $\mathring{\lambda}_1$ by taking a subsequence $\{p_{m_1}\}$ if necessary. For the point sequence $\{p_{m_1}\}$, a sequence $\{\lambda_2(p_{m_1})\}$ is also bounded and hence there is a subsequence $\{p_{m_2}\}$ of $\{p_{m_1}\}$ such that $\{\lambda_2(p_{m_2})\}$ converges to some $\mathring{\lambda}_2$ as m_2 tends to infinity. Thus we can inductively show that there exists a point sequence $\{p_{m'}\}$ of $\{p_m\}$ such that the property (4.4) holds. By (4.3) and (4.4) we get

$$(c + \mathring{\lambda}_i\mathring{\lambda}_j)(\mathring{\lambda}_i - \mathring{\lambda}_j)^2 = 0, \tag{4.5}$$

for any distinct indices i and j. By a simple algebraic calculation it is easily seen that the number of distinct limits in $\{\mathring{\lambda}_i\}$ is at most two

Case 1. If all limits $\mathring{\lambda}_i$ coincide with each other, because $S - nH^2 = \sum \lambda_i^2 - (1/n)(\sum \lambda_i)^2 = (1/n)\sum_{i<j}(\lambda_i - \lambda_j)^2$, it follows from the above property that $\lim_{m_2' \to\infty}(S - nH^2)(p_{m'}) = 0$. Combining this with (4.2), we get $\sup S = nH^2$. Hence the function S becomes a constant nH^2. Accordingly, the hypersurface M^n is totally umbilical.

Case 2. If $\{\mathring{\lambda}_i\}$ has exactly two distinct elements, without loss of generality, we may set

$$\mathring{\lambda}_1 = \cdots = \mathring{\lambda}_k = \lambda, \quad \mathring{\lambda}_{k+1} = \cdots = \mathring{\lambda}_n = \mu, \qquad \lambda < \mu,$$

for some $k = 1, 2, \ldots, n-1$. From (4.5) we have

$$\lambda\mu = -c. \tag{4.6}$$

Now, we only consider the case $c < 0$. As for the cases $c > 0$ and $c = 0$, the proof is simpler and similar to each other, we omit them.

Without loss of generality, we may assume that $H \geq 0$. Then two limits λ and μ are positive. In this case we define a negative number c_1 and a positive number c_2 by $\lambda^2 = c_1 - c$ and $\mu^2 = c_2 - c$, respectively, so these numbers satisfy

$$\begin{cases} c < c_1 < 0, \quad c_2 > 0, \\ \sqrt{(c_1 - c)(c_2 - c)} = -c, \quad \text{i.e., } 1/c_1 + 1/c_2 = 1/c. \end{cases} \tag{4.7}$$

By $H = (1/n)\sum \lambda_i(p_{m'}) = \text{constant}$, we have

$$nH = k\lambda + (n-k)\mu = k\sqrt{c_1 - c} + (n-k)\sqrt{c_2 - c}. \tag{4.8}$$

At the same time we have

$$\begin{aligned} \sup S &= \lim_{m' \to \infty} \sum \lambda_i^2(p_{m'}) = k\lambda^2 + (n-k)\mu^2 \\ &= k(c_1 - c) + (n-k)(c_2 - c). \end{aligned} \tag{4.9}$$

On the other hand, by using (4.7) and (4.8) we can easily see that

$$\begin{aligned} nc + \frac{n^3H^2}{2k(n-k)} &\pm \frac{n(n-2k)}{2k(n-k)}|H|\sqrt{n^2H^2 + 4k(n-k)c} \\ &= k(c_1 - c) + (n-k)(c_2 - c), \end{aligned}$$

where the plus (resp. minus) sign is taken if $k\sqrt{c_1 - c} \geq (n-k)\sqrt{c_2 - c}$ (resp. $k\sqrt{c_1 - c} < (n-k)\sqrt{c_2 - c}$).

Thus, from (4.9) we have

$$\sup S = nc + \frac{n^3H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)}|H|\sqrt{n^2H^2 + 4k(n-k)c}.$$

When $k > 1$, the sectional curvature $K_{12}(p_{m'}) = (c + \lambda_1\lambda_2)(p_{m'}) \underset{m' \to \infty}{\longrightarrow} c + \lambda^2 = c_1 < 0$, so from the assumption we have $k = 1$. Then $k\sqrt{c_1 - c} < (n-k)\sqrt{c_2 - c}$. Therefore we have at last

$$\sup S = nc + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}|H|\sqrt{n^2H^2 + 4(n-1)c}.$$

This completes the proof of the first part of the theorem.

In particular, if $S = \text{constant}$ and M^n is connected, then (2.4) says that all the principal curvatures are constant and that they satisfy

$$(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 = 0, \tag{4.10}$$

for any distinct indices i and j. Hence the number of distinct elements in $\{\lambda_i\}$ is at most two.

If $\lambda_1 = \cdots = \lambda_n$, then M^n is totally umbilical.

If $\{\lambda_i\}$ has exactly two distinct elements, without loss of generality we may assume that $\lambda_1 = \cdots = \lambda_k = \lambda$, $\lambda_{k+1} = \cdots = \lambda_n = \mu$, $\lambda < \mu$, for some $k = 1, 2, \ldots, n-1$. From (4.10) we have

$$\lambda\mu = -c. \tag{4.11}$$

For $c \geq 0$, the theorem has been proved by Nomizu and Smyth [3], so we just consider the case $c < 0$. Without loss of generality we may assume that

$H \geq 0$. Then, similar to the first part, we get $\lambda = \sqrt{c_1 - c}$, $\mu = \sqrt{c_2 - c}$ and $k = 1$, where c_1, c_2 satisfy (4.7). Thus the second fundamental form of M^n in $H^{n+1}(c)$ is given by

$$(h_{ij}) = \text{diag}(\sqrt{c_1 - c}, \sqrt{c_2 - c}, \dots, \sqrt{c_2 - c}).$$

Then, by using the method similar to that of [3] and combining with Section 3, we can show that M^n is isometric to $H^1(c_1) \times S^{n-1}(c_2)$. Q.E.D.

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Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, Henan, People's Republic of China

Current address: Department of Mathematics, Sichuan University, Chengdu, 610064, Sichuan, People's Republic of China