The classical Dvoretzky spherical sections theorem [5, 13] states that $l_2$ is finitely represented in any infinite-dimensional Banach space. Using this, the Riesz Representation theorem (for finite-dimensional $l_p$ spaces) and the Hahn-Banach theorem, we show that $l_\infty$ is finitely represented in $\mathcal{P}(nE)$, for any infinite-dimensional Banach space and any $n \geq 2$. This shows that $\mathcal{P}(nE)$ does not have any non-trivial superproperties and explains why spaces such as Tsirelson’s space play such a positive role in the recent theory of polynomials on Banach spaces ([1, 2, 6, 7, 8, 9, 10]). We refer to [3, 11, 12] for properties of Banach spaces and to [4] for properties of polynomials.

Theorem 1. Suppose $E$ is a Banach space, $1 < p \leq \infty$, $\{\phi_j\}_{j=1}^k$ is a finite sequence of vectors in $E'$ and $A$ and $B$ are positive constants such that

\begin{equation}
A^p \sum_{j=1}^k |\alpha_j|^p \leq \left( \sum_{j=1}^k |\alpha_j \phi_j|^p \right)^{\frac{p}{q}} \leq B^p \sum_{j=1}^k |\alpha_j|^p
\end{equation}

for any sequence of scalars $(\alpha_j)_{j=1}^k$. Then for any integer $n$, $n \geq q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and any sequence of scalars $(\alpha_j)_{j=1}^k$ we have

\begin{equation}
A^n \sup_{1 \leq j \leq k} |\alpha_j| \leq \left( \sum_{j=1}^k |\alpha_j \phi_j|^p \right)^{\frac{n}{q}} \leq B^n \sup_{1 \leq j \leq k} |\alpha_j|.
\end{equation}

Proof. For any $x \in E$, $\|x\| \leq 1$, we have

\begin{equation}
\sup_{\sum_{j=1}^k |\alpha_j|^p \leq 1} \left| \sum_{j=1}^k |\alpha_j \phi_j(x)|^p \right| \leq B^p.
\end{equation}
Since \((l_p)' = l_q\), this implies
\[
\sup_{\|x\| \leq 1} \sum_{j=1}^{k} |\phi_j(x)|^q \leq B^q.
\]

If \(n \geq q\), then
\[
\sup_{\|x\| \leq 1} \left| \sum_{j=1}^{k} \alpha_j \phi_j^n(x) \right| \leq \sup_{\|x\| \leq 1} |\alpha_j| \cdot B^n \sup_{\|x\| \leq 1} \left| \sum_{j=1}^{k} \phi_j(x) \right|^n 
\leq B^n \sup_{\|x\| \leq 1} |\alpha_j|.
\]

On the other hand
\[
A^p \sum_{j=1}^{k} |\alpha_j|^p = A^p \sup_{\|x\| \leq 1} \left| \sum_{j=1}^{k} \alpha_j \beta_j \right|^p \leq \sup_{\|x\| \leq 1} \left| \sum_{j=1}^{k} \alpha_j \phi_j(x) \right|^p.
\]

Since the set \(\{(\phi_j(x))_{j=1}^{k}; \|x\| \leq 1\}\) is a convex balanced set, the Hahn-Banach theorem implies that
\[
A \cdot B_{l_q}^n = A \cdot \left\{ (\beta_j)_{j=1}^{k}; \sum_{j=1}^{k} |\beta_j|^q \leq 1 \right\} \subset \{(\phi_j(x))_{j=1}^{k}; \|x\| \leq 1\}.
\]

Hence, for any fixed integer \(l, 1 \leq l \leq k\), there exists \((x_n)_n\) in \(E, \|x_n\| \leq 1\), such that
\[
\phi_l(x_n) \to A \quad \text{and, for } j \neq l, \quad \phi_j(x_n) \to 0 \quad \text{as } n \to \infty.
\]

This implies that
\[
\sup_{\|x\| \leq 1} \left| \sum_{j=1}^{k} \alpha_j \phi_j^n(x) \right| \geq A^n \cdot \sup_{1 \leq j \leq k} |\alpha_j|,
\]
and the inequalities (3) and (4) prove the proposition.

Note that in the proof of Theorem 1 we have actually shown that the left- (resp. right-) hand side of the inequality (1) implies the left- (resp. right-) hand side of the inequality (2). Conditions (1) and (2) can be rephrased in terms of the Banach-Mazur distance \(d\) to give the following result.

**Corollary 2.** Let \(E\) denote a \(k\)-dimensional Banach space and suppose \(d(E, l_p^n) \leq C\) where \(1 \leq p < \infty\). Then, for \(n \geq p\), \(\mathcal{P}(nE)\) contains a \(k\)-dimensional subspace \(F\) such that
\[
d(F, l_\infty^n) \leq C^n.
\]

**Corollary 3.** If \(E\) is an infinite-dimensional Banach space and \(n \geq 2\), then \(l_\infty^n\) is finitely represented in \(\mathcal{P}(nE)\).

**Proof.** By the classical Dvoretzky theorem we can choose for any positive integer \(k\) and any \(\varepsilon > 0\) vectors \(\{\phi_j\}_{j=1}^{k}\) in \(E'\) such that
\[
\sum_{j=1}^{k} |\alpha_j|^2 \leq \left( \sum_{j=1}^{k} |\alpha_j \phi_j| \right)^2 \leq (1 + \varepsilon)^2 \sum_{j=1}^{k} |\alpha_j|^2
\]
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for any sequence of scalars \((\alpha_j)_{j=1}^k\). Hence, by Theorem 1, we have, for \(n \geq 2\) and any \((\alpha_j)_{j=1}^k\),

\[
\sup_{1 \leq j \leq k} |\alpha_j| \leq \left\| \sum_{j=1}^k \alpha_j \phi_j^* \right\| \leq (1 + \epsilon)^n \sup_{1 \leq j \leq k} |\alpha_j|.
\]

This proves the corollary.

In fact it is easily seen that the above shows that \(l_\infty\) is finitely represented in the space of polynomials of finite type.

**Corollary 4.** If \(l_p\), \(1 \leq p < \infty\), is a quotient of \(E\), then \(l_\infty\) is a subspace of \(\mathcal{P}(nE)\), \(n \geq p\), and \(l_1\) is a complemented subspace of the completed symmetric tensor product endowed with the projective topology, \(\mathfrak{D}_{n,s,\pi} E\).

**Proof.** Let \(1/p + 1/q = 1\). We can choose constants \(A\) and \(B\) independent of \(k\) and vectors \((\phi_k)_k\) in \(E'\) such that

\[
A^q \cdot \sum_{j=1}^k |\alpha_j|^q \leq \left\| \sum_{j=1}^k \alpha_j \phi_j^* \right\|^q \leq B^q \cdot \sum_{j=1}^k |\alpha_j|^q
\]

for any sequence of scalars \((\alpha_j)_j\). Theorem 1 implies that, for \(n \geq p\),

\[
A^n \sup_{1 \leq j \leq k} |\alpha_j| \leq \left\| \sum_{j=1}^k \alpha_j \phi_j^* \right\| \leq B^n \sup_{1 \leq j \leq k} |\alpha_j|
\]

for any sequence of scalars \((\alpha_j)_j\) and any \(k\). Hence \(\{(\phi_j^*)_{j=1}^\infty\}\) is equivalent to the unit vector basis of \(c_0\). Since \(\mathcal{P}(nE)\) is a dual space, this implies \(l_\infty \hookrightarrow \mathcal{P}(nE)\). Since \((\mathfrak{D}_{n,s,\pi} E) = \mathcal{P}(nE)\), it follows by [3, p. 48] that \(l_1\) is a complemented subspace of \(\mathfrak{D}_{n,s,\pi} E\). This completes the proof.

This result for \(p = 2\) is given in [8, Proposition 13] and also implies the well-known fact that \(3d(l_p)\) is not reflexive if \(n \geq p\).

We now extend the result given in Corollary 3 and at the same time obtain a refinement of [6, Theorem 1(ii)]. The elements of \(\mathfrak{D}_{n,s,\epsilon} E'\) are \(n\)-homogeneous polynomials on \(E\) which are uniformly weakly continuous on bounded subsets of \(E\). Hence they have unique extensions to \(E''\). We use the notation \(\tilde{P}\) to denote this extension.

**Lemma 5.** A bounded sequence \((P_j)_j\) in \(\mathfrak{D}_{n,s,\epsilon} E'\) is a weakly null sequence if and only if \(\tilde{P}_j(x'') \to 0\) as \(j \to \infty\) for any \(x'' \in E''\).

**Proof.** If \(\phi \in (\mathfrak{D}_{n,s,\epsilon} E)'\), then there exists a regular Borel measure \(\mu\) on \((\overline{B}_{E''}, \sigma(E'', E'))\) such that

\[
\phi(P) = \int_{\overline{B}_{E''}} \tilde{P}(x'') \, d\mu(x'')
\]

for all \(P \in \mathfrak{D}_{n,s,\epsilon} E'\).

If \((P_j)_j\) is bounded, then \((\tilde{P}_j)_j\) is uniformly bounded on \(\overline{B}_{E''}\). If \(\tilde{P}_j(x'') \to 0\) as \(j \to \infty\) for each \(x'' \in \overline{B}_{E''}\), then the Lebesgue dominated convergence
theorem implies that \( \phi(P_j) \to 0 \) as \( j \to \infty \). Hence \( (P_j)_j \) is weakly null. The converse is obvious.

**Proposition 6.** If \( E \) is a Banach space and \( E' \) contains a weakly null sequence of unit vectors, which satisfies an upper \( q \) estimate, \( q < \infty \), then for \( n \geq p \), \( 1/p + 1/q = 1 \), we have \( \ell_\infty \hookrightarrow \mathcal{P}(^{*}E) \).

**Proof.** It suffices to show \( c_0 \hookrightarrow \mathcal{X}_{n,s,\varepsilon}E' \). Let \( (\phi_j)_j \) denote a weakly null sequence of unit vectors in \( E' \) which satisfies an upper \( q \)-estimate. Then \( (\phi^n_j)_j \) is a sequence of unit vectors in \( \mathcal{X}_{n,s,\varepsilon}E' \). Since \( (\phi_j)_j \) is weakly null, Lemma 5 implies that \( (\phi^n_j)_j \) is a weakly null sequence in \( \mathcal{X}_{n,s,\varepsilon}E' \). By the Bessaga-Pelczynski selection principle [3, p. 42; 11, p. 5] the sequence \( (\phi^n_j)_j \) contains a subsequence which forms a basic sequence. Since upper \( q \) estimates are inherited by subsequences, we may suppose that \( (\phi^n_j)_j \) is a basic sequence.

Hence there exists \( B > 0 \) such that

\[
\left\| \sum_{j=1}^{k} \alpha_j \phi_j \right\|^q \leq B^q \sum_{j=1}^{k} |\alpha_j|^q
\]

for any integer \( k \) and any sequence of scalars \( (\alpha_j)_j \).

By Theorem 1 we have

\[
\left\| \sum_{j=1}^{k} \alpha_j \phi^n_j \right\| \leq B^n \sup_{1 \leq j \leq n} |\alpha_j|
\]

for any sequence of scalars \( (\alpha_j)_j \).

Since \( (\phi^n_j)_j \) is a basic sequence, the closed subspace of \( \mathcal{X}_{n,s,\varepsilon}E' \) spanned by \( (\phi^n_j)_j \) is isomorphic to \( c_0 \). This completes the proof.

Proposition 5 applies in particular to reflexive Banach lattices which satisfy a lower \( p \)-estimate \( 1/p + 1/q = 1 \) ([12]).

**BIBLIOGRAPHY**


Department of Mathematics, University College Dublin, Belfield, Dublin 4, Ireland

E-mail address: sdineen@irlearn.bitnet