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A DVORETZKY THEOREM FOR POLYNOMIALS

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ABSTRACT. We lift upper and lower estimates from linear functionals to n-homogeneous polynomials and using this result show that l_∞ is finitely represented in the space of n-homogeneous polynomials, $n \geq 2$, on any infinite-dimensional Banach space. Refinements are also given.

The classical Dvoretzky spherical sections theorem [5, 13] states that l_2 is finitely represented in any infinite-dimensional Banach space. Using this, the Riesz Representation theorem (for finite-dimensional l_p spaces) and the Hahn-Banach theorem, we show that l_∞ is finitely represented in $\mathscr{P}(^nE)$, for any infinite-dimensional Banach space and any $n \geq 2$. This shows that $\mathscr{P}(^nE)$ does not have any non-trivial superproperties and explains why spaces such as Tsirelson's space play such a positive role in the recent theory of polynomials on Banach spaces ([1, 2, 6, 7, 8, 9, 10]). We refer to [3, 11, 12] for properties of Banach spaces and to [4] for properties of polynomials.

Theorem 1. *Suppose E is a Banach space, $1 < p \leq \infty$, $\{\phi_j\}_{j=1}^k$ is a finite sequence of vectors in E' and A and B are positive constants such that*

$$A^p \sum_{j=1}^k |\alpha_j|^p \leq \left\| \sum_{j=1}^k \alpha_j \phi_j \right\|^p \leq B^p \sum_{j=1}^k |\alpha_j|^p \tag{1}$$

for any sequence of scalars $(\alpha_j)_{j=1}^k$. Then for any integer n, $n \geq q$, where $\frac{1}{p} + \frac{1}{q} = 1$, and any sequence of scalars $(\alpha_j)_{j=1}^k$ we have

$$A^n \sup_{1\leq j\leq k} |\alpha_j| \leq \left\| \sum_{j=1}^k \alpha_j \phi_j^n \right\| \leq B^n \sup_{1\leq j\leq k} |\alpha_j|. \tag{2}$$

Proof. For any $x \in E$, $\|x\| \leq 1$, we have

$$\sup_{\sum_{j=1}^k |\alpha_j|^p \leq 1} \left| \sum_{j=1}^k \alpha_j \phi_j(x) \right|^p \leq B^p.$$

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Since $(l_p)' = l_q$, this implies

$$\sup_{\|x\|\leq 1} \sum_{j=1}^{k} |\phi_j(x)|^q \leq B^q.$$

If $n \geq q$, then

$$\sup_{\|x\|\leq 1} \left| \sum_{j=1}^{k} \alpha_j \phi_j^n(x) \right| \leq \sup_j |\alpha_j| \cdot B^n \sup_{\|x\|\leq 1} \sum_{j=1}^{k} \left| \frac{\phi_j(x)}{B} \right|^n \tag{3}$$
$$\leq B^n \sup_j |\alpha_j|.$$

On the other hand

$$A^p \sum_{j=1}^{k} |\alpha_j|^p = A^p \sup_{\sum_{j=1}^{k} |\beta_j|^q \leq 1} \left| \sum_{j=1}^{k} \alpha_j \beta_j \right|^p \leq \sup_{\|x\|\leq 1} \left| \sum_{j=1}^{k} \alpha_j \phi_j(x) \right|^p.$$

Since the set $\{(\phi_j(x))_{j=1}^k ; \|x\| \leq 1\}$ is a convex balanced set, the Hahn-Banach theorem implies that

$$A \cdot B_{l_q^k} = A \cdot \left\{ (\beta_j)_{j=1}^k ; \sum_{j=1}^{k} |\beta_j|^q \leq 1 \right\} \subset \overline{\{(\phi_j(x))_{j=1}^k ; \|x\| \leq 1\}}.$$

Hence, for any fixed integer l, $1 \leq l \leq k$, there exists $(x_n)_n$ in E, $\|x_n\| \leq 1$, such that

$$\phi_l(x_n) \to A \quad \text{and, for } j \neq l, \quad \phi_j(x_n) \to 0 \quad \text{as } n \to \infty.$$

This implies that

$$\sup_{\|x\|\leq 1} \left| \sum_{j=1}^{k} \alpha_j \phi_j^n(x) \right| \geq A^n \cdot \sup_{1\leq j\leq k} |\alpha_j|, \tag{4}$$

and the inequalities (3) and (4) prove the proposition.

Note that in the proof of Theorem 1 we have actually shown that the left- (resp. right-) hand side of the inequality (1) implies the left- (resp. right-) hand side of the inequality (2). Conditions (1) and (2) can be rephrased in terms of the Banach-Mazur distance d to give the following result.

Corollary 2. *Let E denote a k-dimensional Banach space and suppose $d(E, l_p^k) \leq C$ where $1 \leq p < \infty$. Then, for $n \geq p$, $\mathscr{P}(^nE)$ contains a k-dimensional subspace F such that*

$$d(F, l_\infty^k) \leq C^n.$$

Corollary 3. *If E is an infinite-dimensional Banach space and $n \geq 2$, then l_∞ is finitely represented in $\mathscr{P}(^nE)$.*

Proof. By the classical Dvoretzky theorem we can choose for any positive integer k and any $\varepsilon > 0$ vectors $\{\phi_j\}_{j=1}^k$ in E' such that

$$\sum_{j=1}^{k} |\alpha_j|^2 \leq \left\| \sum_{j=1}^{k} \alpha_j \phi_j \right\|^2 \leq (1+\varepsilon)^2 \sum_{j=1}^{k} |\alpha_j|^2$$

for any sequence of scalars $(\alpha_j)_{j=1}^k$. Hence, by Theorem 1, we have, for $n \geq 2$ and any $(\alpha_j)_{j=1}^k$,

$$\sup_{1\leq j\leq k} |\alpha_j| \leq \left\| \sum_{j=1}^k \alpha_j \phi_j^n \right\| \leq (1+\varepsilon)^n \sup_{1\leq j\leq k} |\alpha_j|.$$

This proves the corollary.

In fact it is easily seen that the above shows that l_∞ is finitely represented in the space of polynomials of finite type.

Corollary 4. *If l_p, $1 \leq p < \infty$, is a quotient of E, then l_∞ is a subspace of $\mathscr{P}(^nE)$, $n \geq p$, and l_1 is a complemented subspace of the completed symmetric tensor product endowed with the projective topology, $\widehat{\bigotimes}_{n,s,\pi} E$.*

Proof. Let $1/p + 1/q = 1$. We can choose constants A and B independent of k and vectors $(\phi_k)_k$ in E' such that

$$A^q \cdot \sum_{j=1}^k |\alpha_j|^q \leq \left\| \sum_{j=1}^k \alpha_j \phi_j \right\|^q \leq B^q \cdot \sum_{j=1}^k |\alpha_j|^q$$

for any sequence of scalars $(\alpha_j)_j$. Theorem 1 implies that, for $n \geq p$,

$$A^n \sup_{1\leq j\leq k} |\alpha_j| \leq \left\| \sum_{j=1}^k \alpha_j \phi_j^n \right\| \leq B^n \sup_{1\leq j\leq k} |\alpha_j|$$

for any sequence of scalars $(\alpha_j)_j$ and any k. Hence $\{\phi_j^n\}_{j=1}^\infty$ is equivalent to the unit vector basis of c_0. Since $\mathscr{P}(^nE)$ is a dual space, this implies $l_\infty \hookrightarrow \mathscr{P}(^nE)$. Since $(\widehat{\bigotimes}_{n,s,\pi} E)' \cong \mathscr{P}(^nE)$, it follows by [3, p. 48] that l_1 is a complemented subspace of $\widehat{\bigotimes}_{n,s,\pi} E$. This completes the proof.

This result for $p = 2$ is given in [8, Proposition 13] and also implies the well-known fact that $\mathscr{P}(^nl_p)$ is not reflexive if $n \geq p$.

We now extend the result given in Corollary 3 and at the same time obtain a refinement of [6, Theorem 1(ii)]. The elements of $\widehat{\bigotimes}_{n,s,\varepsilon} E'$ are n-homogeneous polynomials on E which are uniformly weakly continuous on bounded subsets of E. Hence they have unique extensions to E''. We use the notation $\widetilde{P}$ to denote this extension.

Lemma 5. *A bounded sequence $(P_j)_j$ in $\widehat{\bigotimes}_{n,s,\varepsilon} E'$ is a weakly null sequence if and only if $\widetilde{P}_j(x'') \to 0$ as $j \to \infty$ for any $x'' \in E''$.*

Proof. If $\phi \in (\widehat{\bigotimes}_{n,s,\varepsilon} E')'$, then there exists a regular Borel measure μ on $(\overline{B}_{E''}, \sigma(E'', E'))$ such that

$$\phi(P) = \int_{\overline{B}_{E''}} \widetilde{P}(x'')\, d\mu(x'')$$

for all $P \in \widehat{\bigotimes}_{n,s,\varepsilon} E'$.

If $(P_j)_j$ is bounded, then $(\widetilde{P}_j)_j$ is uniformly bounded on $\overline{B}_{E''}$. If $\widetilde{P}_j(x'') \to 0$ as $j \to \infty$ for each x'' in $\overline{B}_{E''}$, then the Lebesgue dominated convergence theorem implies that $\phi(P_j) \to 0$ as $j \to \infty$. Hence $(P_j)_j$ is weakly null. The converse is obvious.

Proposition 6. *If E is a Banach space and E' contains a weakly null sequence of unit vectors, which satisfies an upper q estimate, $q < \infty$, then for $n \geq p$, $1/p + 1/q = 1$, we have $l_\infty \hookrightarrow \mathscr{P}(^nE)$.*

Proof. It suffices to show $c_0 \hookrightarrow \widehat{\bigotimes}_{n,s,\varepsilon} E'$. Let $(\phi_j)_j$ denote a weakly null sequence of unit vectors in E' which satisfies an upper q-estimate. Then $(\phi_j^n)_j$ is a sequence of unit vectors in $\widehat{\bigotimes}_{n,s,\pi} E'$. Since $(\phi_j)_j$ is weakly null, Lemma 5 implies that $(\phi_j^n)_j$ is a weakly null sequence in $\widehat{\bigotimes}_{n,s,\varepsilon} E'$. By the Bessaga-Pelczynski selection principle [3, p. 42; 11, p. 5] the sequence $(\phi_j^n)_j$ contains a subsequence which forms a basic sequence. Since upper q estimates are inherited by subsequences, we may suppose that $(\phi_j^n)_j$ is a basic sequence.

Hence there exists $B > 0$ such that

$$\left\| \sum_{j=1}^{k} \alpha_j \phi_j \right\|^q \leq B^q \sum_{j=1}^{k} |\alpha_j|^q$$

for any integer k and any sequence of scalars $(\alpha_j)_j$.

By Theorem 1 we have

$$\left\| \sum_{j=1}^{k} \alpha_j \phi_j^n \right\| \leq B^n \sup_{1 \leq j \leq n} |\alpha_j|$$

for any sequence of scalars $(\alpha_j)_j$.

Since $(\phi_j^n)_j$ is a basic sequence, the closed subspace of $\widehat{\bigotimes}_{n,s,\varepsilon} E'$ spanned by $(\phi_j^n)_j$ is isomorphic to c_0. This completes the proof.

Proposition 5 applies in particular to reflexive Banach lattices which satisfy a lower p-estimate $1/p + 1/q = 1$ ([12]).

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