PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 9, September 1995

THE MULTIPLICATIVITY OF THE MINIMAL INDEX OF SIMPLE C^*-ALGEBRAS

SATOSHI KAWAKAMI AND YASUO WATATANI

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We show the multiplicativity of the minimal index for simple C^*-algebras. Although our proof is very short and elementary, it is also valid for subfactors, which was first shown by Kosaki and Longo (1992).

1. INTRODUCTION

The multiplicativity of Jones index [9] for subfactors of type II_1-factors is a basic fact: If $N \subset L \subset M$ are factors of type II_1, then

$$[M : N] = [M : L][L : N] .$$

For an inclusion of infinite factors $N \subset M$, Index E was introduced by Kosaki [12], depending on a normal faithful conditional expectation E of M onto N. Among such conditional expectations, one can choose a unique conditional expectation E_0 such that Index $E_0 \leq$ Index E for any $E : M \to N$ as in Havet [5], Hiai [6] and Longo [14, 15]. This E_0 is called a *minimal conditional expectation*, and Index E_0 is called the *minimal index* of $N \subset M$, often denoted by $[M : N]_0$.

The multiplicativity of the minimal index was shown by Kosaki-Longo [13] in the case of inclusions obtained by basic constructions, reducing to the result in Pimsner-Popa [19, 20]. The general case for subfactors was proved by Longo [16] in his sector theory, applying the above result [13]. H. Kosaki has also informed us that R. Longo had a direct proof free from sector theory [17]. Popa [21] also gives an alternative proof for type II_1-factors. Yamagami [23], Denizeau and Havet [3] have other approaches. Moreover the first-named author has also considered it in a general situation and shown that it is related to the chain rules of indicial derivatives for von Neumann subalgebras [11]. Owing to [6, 7] and [10, 11], the multiplicativity of the minimal index is known to be closely related to the additivity of the relative entropy of Connes-Størmer [2] and Pimsner-Popa [19]. See [4, 18] also for the basic notions of subfactors.

In this note, we shall show the multiplicativity of the minimal index for simple C^*-algebras. Although our proof is very short and elementary, it is also

Received by the editors February 14, 1994.
1991 *Mathematics Subject Classification*. Primary 46L37; Secondary 46L35.

valid for subfactors. We do not require any knowledge on sectors, entropy, nor the Takesaki duality theorem.

2. Index for C^*-subalgebras

We recall some notations and properties on the index for C^*-subalgebras from [22]. Let B be a unital C^*-algebra and A a C^*-subalgebra with the same unit I. Let E be a conditional expectation of B onto A. Throughout this note, conditional expectations are assumed to be faithful. Then, E is called of index-finite type if there exists a finite set $\{u_1, u_2, \dots, u_n\} \subset B$, called a basis for E, such that

$$x = \sum_{i=1}^{n} u_i E(u_i^* x) \quad \text{for any } x \in B.$$

(A finite family $\{(u_1, u_1^*), (u_2, u_2^*), \dots, (u_n, u_n^*)\}$ is a quasi-basis for E in the sense of [22].) When E is of index-finite type, the index of E is defined by

$$\text{Index } E = \sum_{i=1}^{n} u_i u_i^* .$$

The value Index E does not depend on the choice of a basis for E, and Index E is in Center B, the center of B. See Izumi [8] for interesting examples of simple C^*-subalgebras of Cuntz algebras with finite index. When $A \subset B$ is a factor-subfactor pair, Index E coincides with Kosaki's index [12].

Let α be an action of a finite group G on a C^*-algebra A and $B = A \rtimes_\alpha G$ the crossed product. Then, there is a canonical conditional expectation E of B onto A such that

$$E(\sum_{g\in G} a_g \lambda_g) = a_e I \quad \text{for } \sum_{g\in G} a_g \lambda_g \in A \rtimes_\alpha G = B .$$

In this situation, $\{\lambda_g\}$ is a basis for E. If an operator x commutes with A, then $\sum_{g\in G} \lambda_g x \lambda_g^*$ obviously commutes with B because B is generated by A and $\{\lambda_g | g \in G\}$. This fact suggests the following lemma:

Lemma 1. *Let $A \subset B \subset C$ be inclusions of unital C^*-algebras with the same unit and $E : B \to A$ a conditional expectation of index-finite type with a basis $\{u_1, u_2, \dots, u_n\}$ for E. Then, for $x \in A' \cap C$, $\sum_{i=1}^n u_i x u_i^*$ is in $B' \cap C$.*

Proof. For $x \in A' \cap C$, $\sum_{i=1}^n u_i x u_i^*$ commutes with any $b \in B$. Indeed,

$$\begin{aligned} b(\sum_{i=1}^{n} u_i x u_i^*) &= \sum_{i=1}^{n} (b u_i) x u_i^* = \sum_{i=1}^{n}\sum_{j=1}^{n} u_j E(u_j^* b u_i) x u_i^* \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} u_j x E(u_j^* b u_i) u_i^* \quad (\text{since } E(u_j^* b u_i) \in A\ ) \\ &= \sum_{j=1}^{n} u_j x (\sum_{i=1}^{n} E(u_j^* b u_i) u_i^*) = \sum_{j=1}^{n} u_j x (u_j^* b) = (\sum_{i=1}^{n} u_i x u_i^*) b \end{aligned}$$

Therefore, we have $\sum_{i=1}^n u_i x u_i^* \in B' \cap C$. □

Remark A. (1) In the case $C = B$, Lemma 1 asserts that $\sum_{i=1}^{n} u_i x u_i^* \in$ Center B for $x \in A' \cap B$, especially,

$$\text{Index } E = \sum_{i=1}^{n} u_i u_i{*} \in \text{Center } B \quad .$$

(2) In the case $C = B(H)$ for some Hilbert space H, Lemma 1 suggests that

$$F(x) = \sum_{i=1}^{n} u_i x u_i^* \qquad (x \in A')$$

defines a normal bounded operator valued weight F of A' onto B'. Indeed, when $A \subset B$ is a factor-subfactor pair, it is known that $F = E^{-1}$ by [22], [7].

3. Minimal index

We recall that one can minimize indices of conditional expectations of unital C^*-algebras with trivial centers, for example unital simple C^*-algebras.

Proposition 2 [22]. *Let $A \subset B$ be an inclusion of unital C^*-algebras with* Center A = Center $B = \mathbb{C}I$. *Assume that there exists a conditional expectation $F : B \to A$ of index-finite type. Then, there exists a unique minimal conditional expectation $E_0 : B \to A$, i.e.,* Index $E_0 \leq$ Index E *for any conditional expectation $E : B \to A$. Moreover, $E = E_0$ if and only if*

$$\sum_{i=1}^{n} u_i x u_i^* = cE(x) \qquad (x \in A' \cap B)$$

for some constant $c > 0$, where $\{u_1, u_2, \dots, u_n\}$ is a basis for E.

Remark B. (1)The above constant c is given by $c =$ Index E.

(2) Index E_0 is called the minimal index for a pair $A \subset B$ of C^*-algebras and is often denoted by $[B : A]_0$.

(3) When $A \subset B$ is a factor-subfactor pair of finite index, every conditional expectation of B onto A is automatically normal and of finite index. Therefore, observing Remark A(2), we see that the above Proposition 2 is exactly the same as Hiai's characterization of minimal index in [6].

Now we are ready to describe the main theorem, which asserts the multiplicativity of the minimal index for unital simple C^*-algebras.

Theorem 3. *Let $A \subset B \subset C$ be inclusions of unital C^*-algebras with* Center A = Center B = Center $C = \mathbb{C}I$. *Let $E : B \to A$ and $F : C \to B$ be conditional expectations of index-finite type. Then, $E \circ F$ is minimal if and only if E and F are minimal. Moreover, minimal index is multiplicative, that is,*

$$[C : A]_0 = [C : B]_0 \cdot [B : A]_0$$

Proof. Suppose that $E \circ F$ is minimal. Then, the fact that

$$\text{Index}(E \circ F) = (\text{Index } E)(\text{Index } F) \qquad [22, \text{Proposition } 1.7.1]$$

implies that both E and F need to be minimal.

Conversely, suppose that E and F are minimal. Let $\{u_1, u_2, \dots, u_n\} \subset B$ be a basis for E and $\{v_1, v_2, \dots, v_m\} \subset C$ a basis for F. Then, $\{v_j u_i \mid i =$

$1, 2, \ldots, n, j = i, 2, \ldots, m\}$ is a basis for $E \circ F$ by [22, Proposition 1.7.1]. Applying a characterization of the minimal conditional expectation in Proposition 2 to E and F, we have, for $x \in A' \cap C$,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(v_j u_i)x(v_j u_i)^* = \sum_{j=1}^{m} v_j(\sum_{i=1}^{n} u_i x u_i^*)v_j^*$$

$$= (\text{Index } F)F(\sum_{i=1}^{n} u_i x u_i^*) \quad (\text{since } \sum_{i=1}^{n} u_i x u_i^* \in B' \cap C \text{ by Lemma 1})$$

$$= (\text{Index } F)\sum_{i=1}^{n} u_i F(x) u_i^* \quad (\text{since } u_i \in B \text{ and } F : C \to B)$$

$$= (\text{Index } F)(\text{Index } E)E(F(x)) \quad (\text{since } F(x) \in A' \cap B\ ).$$

Using Proposition 2 again for $E \circ F$, we conclude that $E \circ F$ is minimal. The rest is now clear. □

Combining Theorem 3 with Remark B(3), we immediately get the following corollary:

Corollary 4 [11], [16], [21]. *Let $N \subset L \subset M$ be inclusions of factors with finite index in Kosaki's sense. Then, for normal conditional expectations $E : L \to N$ and $F : M \to L$, $E \circ F$ is minimal if and only if E and F are minimal. Moreover,*

$$[M : N]_0 = [M : L]_0 \cdot [L : N]_0.$$

References

1. M. Ballet, Y. Denizeau, and J. F. Havet, *Indice d'une espérance conditionnelle,* Compositio Math. **66** (1988), 199–236.
2. A. Connes and E. Størmer, *Entropy for automorphisms of* II_1*-von Neumann algebras,* Acta Math. **134** (1975), 289–306.
3. Y. Denizeau and J. F. Havet, *Correspondances d'indice fini.* II : *Indice d'une correspondance,* Preprint.
4. F. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Dynkin diagrams and towers of algebras,* Springer-Verlag, Berln, 1989.
5. J. F. Havet, *Espérance conditionnelle minimale,* J. Operator Theory **24** (1990), 33–55.
6. F. Hiai, *Minimizing indices of conditional expectations onto a subfactor,* Publ. Res. Inst. Math. Sci. **24** (1988), 673–678.
7. ———, *Minimum index for subfactors and entropy,* J. Operator Theory **24** (1990), 301–336.
8. M. Izumi, *Subalgebras of infinite* C^**-algebras with finite Watatani indices.* I: *Cuntz algebras,* Comm. Math. Phys. **155** (1993), 157–182.
9. V. F. R. Jones, *Index of subfactors,* Invent. Math. **66** (1983), 1–25.
10. S. Kawakami, *Some remarks on index and entropy for von Neumann subalgebras,* Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), 323–325.
11. ———, *Indicial derivative for von Neumann subalgebras,* preprint.
12. H. Kosaki, *Extension of Jones' theory on index to arbitrary factors,* J. Funct. Anal. **66** (1986), 123–140.
13. H. Kosaki and R. Longo, *A remark on the minimal index of subfactors,* J. Funct. Anal. **107** (1992), 458–470.
14. R. Longo, *Index of subfactors and statistics of quantum fields.* I, Comm. Math. Phys. **126** (1989), 217–247.

15. ———, *Index of subfactors and statistics of quantum fields.* II, Comm. Math. Phys. **130** (1990), 285–309.

16. ———, *Minimal index and braided subfactors,* J. Funct. Anal. **109** (1992), 98–112.

17. ———, *Minimal index and unimodular sectors,* Quantum and Non-commutative Analysis, Kluwer, Dordrecht, 1993, pp. 335–340.

18. A. Ocneanu, *Quantized groups, string algebras, and Galois theory,* Operator Algebras and Applications, Vol. II, Cambridge Univ. Press, Cambridge, 1988, pp. 119–172.

19. M. Pimsner and S. Popa, *Entropy and index of subfactors,* Ann. Sci. École Norm. Sup. (4) **19** (1986), 57–106.

20. ———, *Iterating the basic construction,* Trans. Amer. Math. Soc. **310** (1988), 127–133.

21. S. Popa, *Classification of amenable subfactors of type* II, Acta. Math. **172** (1994), 163–255.

22. Y. Watatani, *Index for C^*-subalgebras,* Mem. Amer. Math. Soc. No. 424 (1990).

23. S. Yamagami, *Modular theory for bimodules,* J. Funct. Anal. **125** (1994), 327–357.

DEPARTMENT OF MATHEMATICS, NARA UNIVERSITY OF EDUCATION, NARA, 630, JAPAN
E-mail address: f61007@sinet.ad.jp

DEPARTMENT OF MATHEMATICS, KYUSHU UNIVERSITY, ROPPONMATU, FUKUOKA, 810, JAPAN
E-mail address: watatani@math.kyushu-u.ac.jp