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ANOTHER GENERALIZATION OF ANDERSON'S THEOREM

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ABSTRACT. In this paper, we prove that if A and B are normal operators on a Hilbert space H, then, for every operator S satisfying $ASB = S$, $\|AXB-X+S\| \geq \|A\|^{-1}\|B\|^{-1}\|S\|$ for all operators $X \in B(H)$, and that if A and B are contractions, then, for every operator S satisfying $ASB = S$ and $A^*SB^* = S$, $\|AXB - X + S\| \geq \|S\|$ for all operators $X \in B(H)$, where $B(H)$ denotes the set of all bounded linear operators on H.

1. INTRODUCTION

Let $B(H)$ denote the set of all bounded linear operators on a Hilbert space H. Anderson ([1]) proved that

Theorem A. *If A is a normal operator, then, for every operator S satisfying $AS = SA$,*

$$\|S - (AX - XA)\| \geq \|S\|$$

for all operators $X \in B(H)$.

Recently, H. Du and W. Xu ([2]) obtained an alternative proof of Anderson's theorem that depends only on the spectral representation of normal operators and proved that

Theorem D-X. *Let operators A and B be in $B(H)$. If an operator C satisfies $AC = CB$, $A^*C = CB^*$ and $\|C\| > \|C\|_e$, then*

$$\|C - (AX - XB)\| \geq \|C\|,$$

for all $X \in B(H)$, where $\|C\|_e$ denotes the essential norm of C.

Duggal ([3]) proved that if A and B are contractions, then $S \in C_2$ and $ASB - S = 0$ imply $\|AXB - X + S\|_2^2 = \|AXB - X\|_2^2 + \|S\|_2^2$ for all $X \in B(H)$, where C_2 denotes the Hilbert-Schmidt class of $B(H)$.

In this note, we shall prove the following theorems:

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Theorem 1. *If A is normal operator in $B(H)$, then, for every operator S satisfying $ASA = S$,*

$$\|AXA - X + S\| \geq \|A\|^{-2}\|S\|,$$

for all $X \in B(H)$.

Theorem 2. *If A is a contraction in $B(H)$, then, for every operator S satisfying $ASA^* = S$ and $A^*SA = S$,*

$$\|AXA - X + S\| \geq \|S\|,$$

for all $X \in B(H)$.

Remark. In the above two theorems, putting $A = I$, it is easy to see that the estimates are sharp.

2. Proof of the theorems

Proof of Theorem 1. If $\|A\| < 1$, since $ASA = S$ implies $S = 0$, it is nothing to prove. So we assume that $\|A\| \geq 1$. In this case, take any α such that $(1-\alpha)\|A\| < 1$, so $\alpha > 1 - \|A\|^{-1}$. Denote $\Delta_\alpha = \{\lambda \in \sigma(A) \colon |\lambda| \leq 1 - \alpha\}$ and by $E(\cdot)$ the spectral measure of A (the spectrum of an operator T is denoted by $\sigma(T)$). Then $E(\Delta_\alpha)H$ reduces A, so A and S have the operator matrix forms

$$A = \begin{pmatrix} A_{1\alpha} & 0 \\ 0 & A_{2\alpha} \end{pmatrix}, \qquad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

with respect to the space decomposition $H = E(\Delta_\alpha)H \oplus E(\Delta'_\alpha)H$, respectively, where $\Delta'_\alpha = \sigma(A)\backslash\Delta_\alpha$. It is easy to see that $\|A_{1\alpha}\| \leq 1 - \alpha < 1$, $\|A_{2\alpha}\| = \|A\|$ and $\sigma(A_{2\alpha}) \subset \overline{\Delta'_\alpha}$, so $A_{2\alpha}$ is invertible on $E(\Delta'_\alpha)H$ and $\|A_{2\alpha}^{-1}\| \leq \frac{1}{1-\alpha}$. By the hypothesis $ASA = S$, we obtain

$$\begin{pmatrix} A_{1\alpha} & 0 \\ 0 & A_{2\alpha} \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} A_{1\alpha} & 0 \\ 0 & A_{2\alpha} \end{pmatrix} = \begin{pmatrix} A_{1\alpha}S_{11}A_{1\alpha} & A_{1\alpha}S_{12}A_{2\alpha} \\ A_{2\alpha}S_{21}A_{1\alpha} & A_{2\alpha}S_{22}A_{2\alpha} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

so

$$A_{1\alpha}S_{11}A_{1\alpha} = S_{11}, \tag{1}$$
$$A_{1\alpha}S_{12}A_{2\alpha} = S_{12}, \tag{2}$$
$$A_{2\alpha}S_{21}A_{1\alpha} = S_{21}, \tag{3}$$
$$A_{2\alpha}S_{22}A_{2\alpha} = S_{22}. \tag{4}$$

From (1), (2), (3) and every positive integer n, we get

$$A_{1\alpha}^n S_{11} A_{1\alpha}^n = S_{11}, \tag{5}$$
$$A_{1\alpha}^n S_{12} A_{2\alpha}^n = S_{12}, \tag{6}$$
$$A_{2\alpha}^n S_{21} A_{1\alpha}^n = S_{21}, \tag{7}$$

respectively. Therefore,

$$\|A_{1\alpha}\|^{2n}\|S_{11}\| \geq \|S_{11}\|, \tag{8}$$
$$\|A_{1\alpha}\|^n\|S_{12}\|\,\|A_{2\alpha}\|^n \geq \|S_{12}\|, \tag{9}$$
$$\|A_{2\alpha}\|^n\|S_{21}\|\,\|A_{1\alpha}\|^n \geq \|S_{21}\|. \tag{10}$$

But, by the choice of α, $\|A_{1\alpha}\|^n \to 0$ and $(\|A_{1\alpha}\|^n\|A_{2\alpha}\|^n) \le ((1-\alpha)\|A\|)^n \to 0$ (as $n \to \infty$), hence $S_{11} = 0$, $S_{12} = 0$ and $S_{21} = 0$. It shows that

$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_{22} \end{pmatrix}$$

and $\|S_{22}\| = \|S\|$. Letting $X_\alpha = E(\Delta'_\alpha)XE(\Delta'_\alpha)$, we now get

$$\begin{aligned}\|AXA - X + S\| &\ge \|E(\Delta'_\alpha)(AXA - X + S)E(\Delta'_\alpha)\| \\ &= \|A_{2\alpha}X_\alpha A_{2\alpha} - X_\alpha + S_{22}\| \\ &= \|A_{2\alpha}(X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22})\| \\ &\ge \frac{1}{\|A_{2\alpha}^{-1}\|}\|X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22}\| \\ &\ge (1-\alpha)\|X_\alpha A_{1\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22}\|.\end{aligned}$$

Note that by (4) $A_{2\alpha}^{-1}S_{22}A_{2\alpha} - A_{2\alpha}^{-1}A_{2\alpha}^{-1}S_{22} = A_{2\alpha}^{-2}(A_{2\alpha}S_{22}A_{2\alpha} - S_{22}) = 0$, so moreover by Anderson's Theorem

$$\begin{aligned}\|X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22}\| &\ge \|A_{2\alpha}^{-1}S_{22}\| \\ &\ge \frac{1}{\|A_{2\alpha}\|}\|S_{22}\| = \|A\|^{-1}\|S\|.\end{aligned}$$

That is, $\|AXA - X + S\| \ge (1-\alpha)\|A\|^{-1}\|S\|$. But, by the choice of α, we may choose α such that $(1-\alpha)\|A\|$ is sufficiently close to 1, so

$$\|AXA - X + S\| \ge \|A\|^{-2}\|S\|.$$

We have finished the proof.

Corollary 2.1. *If A and B are normal, then for every operator S satisfying $ASB = S$ and all operators $X \in B(H)$,*

$$\|AXB - X + S\| \ge \|A\|^{-1}\|B\|^{-1}\|S\|.$$

Proof. Suppose that

$$\widetilde{A} = \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}}A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}}B \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \widetilde{S} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}\widetilde{A}\widetilde{S}\widetilde{A} &= \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}}A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}}B \end{pmatrix}\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}}A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}}B \end{pmatrix} \\ &= \begin{pmatrix} 0 & ASB \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} = \widetilde{S},\end{aligned}$$

$\|\widetilde{S}\| = \|S\|$, and $\|\widetilde{A}\| = \sqrt{\|A\|\,\|B\|}$. By Theorem 1,

$$\begin{aligned}\|AXB - X + S\| &= \|\widetilde{A}\widetilde{X}\widetilde{A} - \widetilde{X} + \widetilde{S}\| \\ &\ge \|\widetilde{A}\|^{-2}\|\widetilde{S}\| = \|A\|^{-1}\|B\|^{-1}\|S\|.\end{aligned}$$

Now we turn to the proof of Theorem 2.

Proof of Theorem 2. If $\|A\| < 1$, it is clear that $S = 0$, so there is nothing to do. Now we suppose $\|A\| = 1$. For convenience, we will divide the proof into several steps.

(1) Let $A = UP$ be the polar decomposition of A, we shall show that $PS = S = SP$. From the hypotheses $ASA^* = S$ and $A^*SA = S$, we get $A^*ASA^*A = S$; so

$$P^2SP^2 = S.$$

Hence for all positive integers n,

$$P^{2n}SP^{2n} = S.$$

If the spectral measure of P is denoted by $E(\cdot)$, let $\Delta_\varepsilon = \{\lambda \in \sigma(P) : \lambda < 1-\varepsilon\}$ for arbitrary $\varepsilon > 0$; multiplying both sides of the above equation on the left by $E(\Delta_\varepsilon)$, we get

$$(E(\Delta_\varepsilon)PE(\Delta_\varepsilon))^{2n}SP^{2n} = E(\Delta_\varepsilon)S.$$

Since $\|E(\Delta_\varepsilon)PE(\Delta_\varepsilon)\| \le 1-\varepsilon$, it follows that $\lim_{n\to\infty}(E(\Delta_\varepsilon)PE(\Delta_\varepsilon))^{2n} = 0$, so

$$E(\Delta_\varepsilon)S = 0.$$

Similarly, $SE(\Delta_\varepsilon) = 0$. We shall show that $PS = S$ and $SP = S$. In fact, it is clear that the spectrum $\sigma(P)$ is included in $[0, 1]$, so we can suppose that $P = \int_0^1 \lambda\, dE_\lambda$ is the spectral representation of P. Then for any $1 > \varepsilon > 0$, since $\int_0^{1-\varepsilon} \lambda\, dE_\lambda S = 0$, we obtain

$$\begin{aligned}\|PS - S\| &= \left\| \int_0^1 (\lambda - 1)\, dE_\lambda S \right\| \\ &= \left\| \int_{1-\varepsilon}^1 (\lambda - 1)\, dE_\lambda S + \int_0^{1-\varepsilon} (\lambda - 1)\, dE_\lambda S \right\| \le \varepsilon\|S\|.\end{aligned}$$

Because ε is arbitrary, $PS = S$. Similarly, $SP = S$.

(2) We will prove that $SU = US$ and $SU^* = U^*S$.

Denote the range of an operator T by $R(T)$, note that $A^*SA = S$ implies that $R(S) \subset R(A^*)$ and U^*U is the projection on $R(A^*)^-$, where the $R(A^*)^-$ means the closure of $R(A^*)$. So, since $ASA^* = UPSPU^* = USU^*$ from (1) and multiplying both sides of $ASA^* = S$ on the left by U^*, it follows that $U^*S = SU^*$. Similarly, we have $US = SU$.

(3) Suppose that $S = VQ$ is the polar decomposition of S, where $Q = (S^*S)^{-\frac{1}{2}}$, and $\|S\|$ is an isolated point of $\sigma(Q)$. We shall show that in this case the theorem holds.

Since $S^*SA = AS^*A^*SA = AS^*S$, so $QA = AQ$. Let $Q = \int_0^{\|S\|} \lambda\, dF_\lambda$ be the spectral representation of Q; by the hypothesis that $\|S\|$ is an isolated point of $\sigma(Q)$, then $F(\|S\|)A = AF(\|S\|)$. From (1), we get $A^*S^*SA = S^*S$, that is, $A^*Q^2A = Q^2$. Multiplying both sides of the above on the left and the right by $F(\|S\|)$, respectively, and defining $A_1 = F(\|S\|)AF(\|S\|)$, then

$$A_1^*\|S\|^2A_1 = \|S\|^2.$$

We therefore obtain $A_1^*A_1 = I$ on the space $F(\|S\|)H$, where I denotes the identity on $F(\|S\|)H$. Similarly, $A_1A_1^* = I$. These show that A_1 is a unitary operator on the space $F(\|S\|)H$. In this case,

$$\begin{aligned}
&\|S\|\,\|AXA^* - X + S\| \\
&\quad \geq \|S^*AXA^* - S^*X + S^*S\| \\
&\quad = \|AS^*XA^* - S^*X + S^*S\| \\
&\quad \geq \|F(\|S\|)(AS^*XA^* - S^*X + S^*S)F(\|S\|)\| \\
&\quad = \|A_1F(\|S\|)S^*XF(\|S\|)A_1^* - F(\|S\|)S^*XF(\|S\|) + \|S\|^2F(\|S\|)\|.
\end{aligned}$$

Let $X_1 = F(\|S\|)S^*XF(\|S\|)$ and note that A_1 is a unitary operator on $F(\|S\|)H$; then

$$\begin{aligned}
&\|A_1F(\|S\|)S^*XF(\|S\|)A_1^* - F(\|S\|)S^*XF(\|S\|) + \|S\|^2F(\|S\|)\| \\
&\quad = \|A_1X_1 - X_1A_1 + \|S\|^2A_1\| \quad \text{(by Anderson's theorem)} \\
&\quad \geq \|S\|^2\|A_1\| = \|S\|^2,
\end{aligned}$$

so

$$\|AXA^* - X + S\| \geq \|S\|.$$

(4) *The general case.* As in case (3), let $S = VQ$ be the polar decomposition of S. For any $\varepsilon > 0$, define $Q_\varepsilon = \int_0^{1-\varepsilon} \lambda\, dF - \lambda + F([\|S\| - \varepsilon, \|S\|])$ and $S_\varepsilon = VQ_\varepsilon$; since $QA = AQ$, we get $F([\|S\| - \varepsilon, \|S\|])A = AF([\|S\| - \varepsilon, \|S\|])$ and

$$\begin{aligned}
\int_0^{\|S\|-\varepsilon} \lambda\, dF_\lambda A &= F([0, \|S\| - \varepsilon]) \int_0^{\|S\|} dF_\lambda A = F([0, \|S\| - \varepsilon])AQ \\
&= AF([0, \|S\| - \varepsilon])Q = A\int_0^{\|S\|-\varepsilon} \lambda\, dF - \lambda.
\end{aligned}$$

So $Q_\varepsilon A = AQ_\varepsilon$.

Next, from $ASA^* = S$ follows $AVQA^* = VQ$, hence $(AVA^* - V)Q = 0$. It is clear that $R(Q_\varepsilon) = R(Q)$; therefore $(AVA^* - V)Q_\varepsilon = 0$, that is, $AS_\varepsilon A^* = S_\varepsilon$. Similarly, $A^*S_\varepsilon A = S_\varepsilon$. Clearly, $\|S\| = \|S_\varepsilon\|$. By (3), we get

$$\begin{aligned}
\|AXA^* - X + S\| &= \|AXA^* - X + S_\varepsilon - S_\varepsilon + S\| \\
&\geq \|AXA^* - X + S_\varepsilon\| - \|S_\varepsilon - S\| \\
&= \|S_\varepsilon\| - \varepsilon = \|S\| - \varepsilon.
\end{aligned}$$

Finally, since ε is arbitrary,

$$\|AXA^* - X + S\| \geq \|S\|.$$

The proof is completed.

Corollary 2.2. *Let A and B be operators in $B(H)$. If $\|A\|\,\|B\| \leq 1$, then, for every operator S satisfying $ASB = S$ and $A^*SB^* = S$,*

$$\|AXB - X + S\| \geq \|S\|,$$

for all $X \in B(H)$.

Proof. Define

$$\widetilde{A} = \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}}A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}}B^* \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \widetilde{S} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$

Then $\widetilde{A}\widetilde{S}\widetilde{A}^* = \widetilde{S}$ and $\widetilde{A}^*\widetilde{S}\widetilde{A} = \widetilde{S}$; by Theorem 2

$$\begin{aligned}\|AXB - X + S\| &= \|\widetilde{A}\widetilde{X}\widetilde{A}^* - \widetilde{X} + \widetilde{S}\| \\ &\geq \|\widetilde{S}\| = \|S\|.\end{aligned}$$

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