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MONOTONICITY PROPERTIES OF LORENTZ SPACES

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ABSTRACT. Criteria for uniform monotonicity, local uniform monotonicity and strict monotonicity of Lorentz spaces are given.

Let L^0 denote the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0, \gamma)$, $\gamma \le \infty$. In what follows, if $f, g \in L^0$, then $f \le g$ means $f(t) \le g(t)$ almost everywhere (a.e.) with respect to the Lebesgue measure m on the real line. If $f \in L^0$ we denote by d_f the distribution function of $|f|$, that is,

$$d_f(t) = m\{s : |f(s)| > t\},$$

and we denote by f^* the decreasing rearrangement of $|f|$, that is,

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}.$$

Let $w : [0, \gamma) \to \mathbb{R}_+$, $\gamma \le \infty$, be a nonincreasing, strictly positive, locally integrable function with respect to the Lebesgue measure m, called a weight function. Then the Lorentz space Λ_w is defined as follows (see [5,8,11,13])

$$\Lambda_w = \{f \in L^0 : \|f\| = \int_0^\gamma f^*(t)w(t)\,dt = \int_0^\gamma f^* w < \infty\}.$$

By $S(t)$ denote the function $S(t) = \int_0^t w(s)\,ds$. The weight $w(t)$ is called *regular* [5,8,10] if $\inf_{t\in(0,\gamma)} \frac{S(2t)}{S(t)} > 1$.

Let $(E, \|\ \|_E)$ be a Banach function lattice over the measure space (T, Σ, μ) [2,13,14]. E is said to be *strictly monotone* if for every $x, y \in E^+$ (the positive part of E), $x \ge y$, $\|x\|_E = 1$ and $y \ne 0$ imply that $\|x - y\|_E < 1$. Following Birkhoff [2] we say that E is *uniformly monotone* if for any $\epsilon \in (0, 1)$ there exists $\delta(\epsilon) \in (0, 1)$ such that $x, y \in E^+$, $y \le x$, $\|x\|_E = 1$ and $\|y\| \ge \epsilon$ imply $\|x - y\|_E \le 1 - \delta(\epsilon)$. The space E is called *locally uniformly monotone* if for any $\epsilon \in (0, 1)$ and $x \in E^+$ with $\|x\| = 1$, there exists $\delta(\epsilon, x) \in (0, 1)$ such that $\|x - y\|_E \le 1 - \delta(\epsilon, x)$ whenever $y \in E^+$, $y \le x$ and $\|y\|_E \ge \epsilon$.

Obviously uniform monotonicity implies local uniform monotonicity and local uniform monotonicity implies strict monotonicity.

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Monotonicity properties play in Banach lattices the analogous role as rotundity in Banach spaces. They are applicable among others to the dominated best approximation problems (see e.g. [4,9,12]) and to ergodic theory (see [1]). Those properties, with some of their applications, for Orlicz spaces for both the Orlicz and the Luxemburg norm were considered by Bru and Heinich in [3,4], and by Medzhitov and Sukochev in [15]. In Musielak-Orlicz spaces, they were studied by Kurc in [12] and Hudzik and Kurc in [9]. Recently, the monotonicity properties of Calderon-Lozanovskii spaces were investigated in [6].

It is known that any uniformly convex Banach lattice E is uniformly monotone. The Lorentz spaces $\Lambda_{p,w}$ $(1 < p < \infty)$, defined as the set of those $f \in L^0$ for which $\|f\|^p_{\Lambda_{p,w}} = \int_0^\gamma (f^*)^p w < \infty$, are uniformly convex and so uniformly monotone as well, if the weight function w is regular (see [8]). This result was extended later to Orlicz-Lorentz spaces in [10]. For $p = 1$, $\Lambda_{1,w} \equiv \Lambda_w$ is not strictly convex. However, the space L^1 (a particular case of Λ_w when $w(t) \equiv 1$) is uniformly monotone, being flat with no extreme points. In this paper we show that Λ_w is uniformly monotone whenever w is a regular function. The necessity of regularity of w is also proven. Strict and local uniform monotonicities of Λ_w are also studied.

The theorems stated below are the main results of the paper.

Theorem 1. *The Lorentz space Λ_w is uniformly monotone if and only if the weight function w is regular.*

Theorem 2. *Λ_w is locally uniformly monotone whenever $\gamma < \infty$. If $\gamma = \infty$ and $\int_0^\gamma w < \infty$, then Λ_w is not strictly monotone.*

The proof of Theorem 1 will be preceded by two lemmas.

Lemma 1. *Let the weight w be a regular function. Then for every $p \in \{0\} \cup \mathbb{N}$ there exists a constant $k(p) > 0$ such that*

$$S(2^p a + t) - S(2^p a) \geq k(p) S(t)$$

for every $t \geq a$. If $t < a$ and $2^p t \geq a$ for some $p \in \{0\} \cup \mathbb{N}$, then

$$S(t + a) - S(a) \geq k(p) S(t).$$

Proof. If $t \geq a$, then by monotonicity of w,

$$S(2^p a + t) - S(2^p a) \geq S((2^p + 1)t) - S(2^p t).$$

Moreover, by regularity of the weight,

$$S(2^{p+1} t) - S(2^p t) \geq (k-1) S(2^p t) \geq (k-1)^{p+1} S(t)$$

and

$$2^p (S((2^p + 1)t) - S(2^p t)) \geq S(2^{p+1} t) - S(2^p t) \geq (k-1)^{p+1} S(t).$$

This implies that $S(2^p a + t) - S(2^p a) \geq k(p) S(t)$ with $k(p) = 2(\frac{k-1}{2})^{p+1}$.

If $t < a$ and $2^p t \geq a$, then $S(a+t) - S(a) \geq S((2^p+1)t) - S(2^p t) \geq k(p) S(t)$.

Lemma 2. *Let the weight w be a regular function. Then for any $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that for any simple function with compact support and any measurable set A, if $\|f\| \leq 1$ and $\|f\chi_A\| \geq \epsilon$, then*

$$\|f - f\chi_A\| \leq \|f\| - \eta(\epsilon).$$

Proof. Let f be a simple function and A a measurable set such that $\|f\| \le 1$ and $\|f\chi_A\| \ge \epsilon$. In the first part of the proof assume that f is nonincreasing, that is,

$$f = \sum_{i=1}^{n} \alpha_i \chi_{[a_{i-1}, a_i]},$$

where $\alpha_1 > \alpha_2 > \dots > \alpha_n > 0 = \alpha_0$ and $0 = a_0 < a_1 < \dots < a_n$. Thus

$$\|f\| = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) S(a_i) \le 1.$$

Let $A_i = [a_{i-1}, a_i] \cap A$ for $i = 1, 2, \dots, n$ and $A_0 = \varnothing$. Denote

$$c_i = \sum_{j=1}^{i} mA_j, \qquad i = 0, 1, \dots, n,$$

$$d_i = \sum_{j=0}^{i} m([a_{j-1}, a_j] \setminus A_j), \quad i = 1, 2, \dots, n, \quad d_0 = 0.$$

Notice that $d_i + c_i = a_i$. By the assumption that $\|f\chi_A\| \ge \epsilon$,

$$\begin{aligned} \int (f\chi_A)^* w &= \int \sum_{i=1}^{n} \alpha_i \chi_{[c_{i-1}, c_i]} w \\ &= \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) S(c_i) \\ &\ge \epsilon. \end{aligned} \tag{1}$$

We have $(f - f\chi_A)^* = \sum_{i=1}^{n} \alpha_i \chi_{[d_{i-1}, d_i]}$. Setting $F = f - (f - f\chi_A)^*$,

$$\int F w = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1})(S(a_i) - S(d_i)). \tag{2}$$

Choose $p \in \mathbb{N}$ such that

$$\frac{1}{k^p} \le \frac{\epsilon}{4}, \tag{3}$$

where k is the constant from the regularity condition of w. Define the following subsets of natural numbers. Let

$$N_1 = \{i = 1, \dots, n : 2^p d_i < c_i\} \text{ and } N_0 = \{i = 1, \dots, n : 2^p d_i \ge c_i\}.$$

Then let

$$\begin{aligned} N_2 &= \{i \in N_0 : c_i \ge d_i\}, \\ N_3 &= \{i \in N_0 : 2^p c_i < d_i\}, \\ N_4 &= \{i \in N_0 : c_i < d_i \text{ and } 2^p c_i \ge d_i\}. \end{aligned}$$

The sets N_i $(i = 1, \dots, 4)$ are disjoint, and its union is equal to the set $\{1, 2, \dots, n\}$. Since $\|f\| \le 1$, $a_i \ge c_i$ and w is regular,

$$\begin{aligned} 1 &\ge \sum_{N_1} (\alpha_i - \alpha_{i-1}) S(a_i) \ge \sum_{N_1} (\alpha_i - \alpha_{i-1}) S(c_i) \\ &\ge \sum_{N_1} (\alpha_i - \alpha_{i-1}) S(2^p d_i) \ge k^p \sum_{N_1} (\alpha_i - \alpha_{i-1}) S(d_i), \end{aligned}$$

which implies that

$$\sum_{N_1}(\alpha_i - \alpha_{i-1})S(d_i) \le \frac{1}{k^p} < \frac{\epsilon}{4}. \tag{4}$$

Since $c_i \ge d_i$ for $i \in \mathbb{N}_2$, and $a_i = c_i + d_i$, applying Lemma 1 to $a = d_i$ and $x = c_i$, we get

$$\sum_{N_2}(\alpha_i - \alpha_{i-1})(S(a_i) - S(d_i)) \ge k(p)\sum_{N_2}(\alpha_i - \alpha_{i-1})S(c_i). \tag{5}$$

For $i \in N_3$, $c_i < \frac{1}{2^p}d_i$ and, by the regularity of w, $S(\frac{1}{2^p}d_i) \le \frac{1}{k^p}S(d_i)$. Thus

$$\begin{aligned}\sum_{N_3}(\alpha_i - \alpha_{i-1})S(c_i) &\le \sum_{N_3}(\alpha_i - \alpha_{i-1})S(\frac{1}{2^p}d_i)\\ &\le \frac{1}{k^p}\sum_{N_3}(\alpha_i - \alpha_{i-1})S(d_i)\\ &\le \frac{1}{k^p} < \frac{\epsilon}{4}.\end{aligned} \tag{6}$$

Finally, since $c_i < d_i$ and $2^p c_i \ge d_i$ for $i \in N_4$, by Lemma 1, we get

$$\sum_{N_4}(\alpha_i - \alpha_{i-1})(S(a_i) - S(d_i)) \ge k(p)S(c_i). \tag{7}$$

By virtue of (1), either $\sum_{N_0}(\alpha_i - \alpha_{i-1})S(c_i) \ge \frac{\epsilon}{2}$ or $\sum_{N_1}(\alpha_i - \alpha_{i-1})S(c_i) \ge \frac{\epsilon}{2}$. Assuming the second inequality, by (2) and (4), we get the estimation

$$\begin{aligned}\int Fw &\ge \sum_{N_1}(\alpha_i - \alpha_{i-1})S(a_i) - \sum_{N_1}(\alpha_i - \alpha_{i-1})S(d_i)\\ &\ge \sum_{N_1}(\alpha_i - \alpha_{i-1})S(c_i) - \sum_{N_1}(\alpha_i - \alpha_{i-1})S(d_i)\\ &\ge \frac{\epsilon}{2} - \frac{\epsilon}{4} = \frac{\epsilon}{4}.\end{aligned}$$

Now assume that $\sum_{N_0}(\alpha_i - \alpha_{i-1})(S(c_i) \ge \frac{\epsilon}{2}$. Then, by (6),

$$\frac{\epsilon}{2} \le \sum_{N_2 \cup N_4}(\alpha_i - \alpha_{i-1})S(c_i) + \frac{\epsilon}{4},$$

whence

$$\sum_{N_2 \cup N_4}(\alpha_i - \alpha_{i-1})S(c_i) \ge \frac{\epsilon}{4}.$$

This combined with (5) and (7) imply that

$$\begin{aligned}\int Fw &\ge \sum_{N_2 \cup N_4}(\alpha_i - \alpha_{i-1})(S(a_i) - S(d_i))\\ &\ge k(p)\sum_{N_2 \cup N_4}(\alpha_i - \alpha_{i-1})S(c_i)\\ &\ge k(p)\frac{\epsilon}{4}.\end{aligned}$$

Thus we showed that $\int Fw \geq k(p)\frac{\epsilon}{4}$. Hence

$$\|f - f\chi_A\| = \int (f - f\chi_A)^* w = \int fw - \int Fw \leq \|f\| - k(p)\frac{\epsilon}{4}.$$

The constant $\eta(\epsilon) = k(p)\frac{\epsilon}{4}$ depends only on ϵ.

Now let f be any simple function with compact support satisfying the assumptions of the lemma. There exists a measure-preserving transformation $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f^*(t) = f(\sigma(t))$. Thus $(f - f\chi_A)^* = (f(\sigma) - f(\sigma)\chi_{\sigma^{-1}(A)})^* = (f^* - f^*\chi_{\sigma^{-1}(A)})^*$ and $\|f\chi_A\| = \int (f^*\chi_{\sigma^{-1}(A)})^* w = \|f^*\chi_{\sigma^{-1}(A)}\|$. Since f^* is a nonincreasing simple function with $\|f^*\chi_{\sigma^{-1}(A)}\| \geq \epsilon$, by the first part of the proof, there exists $\eta(\epsilon) > 0$ such that $\|f^* - f^*\chi_{\sigma^{-1}(A)}\| \leq \|f^*\| - \eta(\epsilon) = \|f\| - \eta(\epsilon)$. Now, the equality $\|f - f\chi_A\| = \|f^* - f^*\chi_{\sigma^{-1}(A)}\|$ completes the proof of the lemma.

Proof of Theorem 1. Assume that w is a regular function, and let f, g be simple functions with compact supports such that $\|f\| \leq 1$, $0 \leq g \leq f$ and $\|g\| \geq \epsilon$. Define

$$A = \{t \in \mathbb{R}_+ : g(t) \leq \frac{\epsilon}{2} f(t)\}.$$

Then $\epsilon \leq \|g\| \leq \frac{\epsilon}{2} + \|g\chi_{A^c}\|$ (A^c is a complement to A), whence $\|f\chi_{A^c}\| \geq \|g\chi_{A^c}\| \geq \frac{\epsilon}{2}$. Moreover,

$$f(t) - g(t) \leq f(t)\chi_A(t) + (f(t) - \frac{\epsilon}{2} f(t))\chi_{A^c}(t) = f(t) - \frac{\epsilon}{2} f(t)\chi_{A^c}(t).$$

By Lemma 2, $\|f - f\chi_{A^c}\| \leq \|f\| - \eta(\frac{\epsilon}{2})$. Thus

$$\begin{aligned} \|f - g\| &\leq \|f - \frac{\epsilon}{2} f\chi_{A^c}\| \leq (1 - \frac{\epsilon}{2})\|f\| + \frac{\epsilon}{2}\|f - f\chi_{A^c}\| \\ &\leq (1 - \frac{\epsilon}{2}) + \frac{\epsilon}{2}(1 - \eta(\frac{\epsilon}{2})) = 1 - \frac{\epsilon}{2}\eta(\frac{\epsilon}{2}). \end{aligned}$$

Finally, since the set of simple functions with compact supports is dense in Λ_w, the norm is uniformly monotone.

Now we shall show that the weight w must be regular if Λ_w is uniformly monotone. For any $u > 0$, choose $a > 0$ such that $\|a\chi_{[0,2u]}\| = aS(2u) = 1$. Setting $f = a\chi_{[0,2u]}$ and $g = a\chi_{[u,2u]}$ we have $0 \leq g \leq f$, $\|f\| = 1$ and $\|g\| = aS(u) \geq \frac{1}{2}aS(2u) = \frac{1}{2}$. By uniform monotonicity, there exists $\eta > 0$ such that $\|f - g\| \leq 1 - \eta$. However, $\|f - g\| = aS(u)$ and so $aS(u) \leq 1 - \eta = (1 - \eta)aS(2u)$. Hence $S(u) \leq (1 - \eta)S(2u)$, which implies that $S(2u) \geq kS(u)$, with $k = \frac{1}{1-\eta} > 1$. Thus w is regular and the proof is finished.

Proof of Theorem 2. If $\int_0^\infty w < \infty$, then taking $f \equiv 1$ and $g = \chi_{[1,\infty)}$, both functions belong to Λ_w, $f > g$ but $f^* = g^* \equiv 1$, which imply that Λ_w is not strictly monotone.

Now assume that $\gamma < \infty$. We will show that Λ_w is locally uniformly monotone. Here it will be convenient to use an equivalent sequence definition of local uniform monotonicity. Assuming $0 \leq f_n \leq f \in \Lambda_w$, $\|f\| = 1$ and $\|f_n\| \to \|f\| = 1$ we need to show that $\|f - f_n\| \to 0$. At first we will prove that $\|f^* - f_n^*\| \to 0$. Indeed $\int_0^\gamma (f^* - f_n^*)w = \|f\| - \|f_n\| \to 0$, which in turn implies that there exists a subsequence (n_k) of natural numbers such that $f_{n_k}^* \to f^*$

a.e. Since $d_{f^*_{n_k}}(t) \leq d_{f^*}(t) < \infty$ for every $t > 0$, we get $(f^* - f^*_{n_k})^*(t) \to 0$ for any $t > 0$, by Proposition 12^0 in [11, Chapter II, Paragraph 2]. Moreover, $(f^* - f^*_{n_k})^* w \leq 2f^* w$ and $\int_0^\gamma f^* w < \infty$. Therefore, by the Lebesgue dominated theorem, $\|f^* - f^*_{n_k}\| = \int_0^\gamma (f^* - f^*_{n_k})^* w \to 0$.

Now, by a renorming result in [7], there exists a symmetric equivalent norm $\| \|_0$ in Λ_w that is locally uniformly rotund (for definition of local uniform rotundity, see e.g. [13]). Set $\alpha = \frac{1}{\|f\|_0}$. By the first part of the proof, since $0 \leq \frac{f_n+f}{2} \leq f$ and $\|\frac{f_n+f}{2}\| \to \|f\| = 1$, we get $\|(\frac{f_n+f}{2})^* - f^*\| \to 0$. Therefore $\|\alpha(\frac{f_n+f}{2})^* - \alpha f^*\|_0 \to 0$. Consequently $\|\alpha f_n + \alpha f\|_0 = \|\alpha(f_n + f)^*\|_0 \to 2\alpha\|f^*\|_0 = 2$. Since $\|\alpha f_n\|_0 \leq \|\alpha f\|_0 = 1$ and $\| \|_0$ is locally uniformly rotund, $\|\alpha f_n - \alpha f\|_0 \to 0$ and equivalently $\|f_n - f\| \to 0$, which completes the proof of theorem.

Remarks. The growth condition defining regularity of w can be expressed equivalently as follows. For every $l > 1$ there exists $k(l) > 1$ such that $S(lt) \geq k(l)\,S(t)$ for all $t \geq 0$. Indeed, for $l \in (1, 2)$, $\int_t^{2t} w \leq \frac{1}{l-1}\int_t^{lt} w$ by monotonocity of w. Assuming that $S(2t) \geq k\,S(t)$,

$$\begin{aligned} S(lt) &= S(t) + \int_t^{lt} w \geq S(t) + (l-1)\int_t^{2t} w \\ &= S(t) + (l-1)(S(2t) - S(t)) \geq S(t) + (l-1)(kS(t) - S(t)) \\ &= k(l)\,S(t), \end{aligned}$$

with $k(l) = (1 + (l-1)(k-1)) > 1$. The opposite implication is obvious.

One can ask whether it is possible to express the regularity of w in terms of the function w itself. We can easily show that if w is regular, then there exists a constant $l \in (0, 1)$ such that $w(2t) \geq l\,w(t)$ for all $t \in \mathbb{R}_+$. Moreover, if w satisfies the last inequality with $l \in (\frac{1}{2}, 1)$, then w is regular. However, there are examples of functions that are not regular but satisfy the inequality $w(2t) \geq \frac{1}{2}w(t)$ for all $t \in \mathbb{R}_+$. For instance, take $w(t) = 1$ for $t \in [0, 1]$ and $w(t) = \frac{1}{t}$ when $t > 1$.

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