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A NON-HOMOGENEITY PROPERTY OF σ-IDEALS

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ABSTRACT. The object of this note is to prove a theorem of P. Chernoff and R. Solovay concerning a structural property of ω_1-saturated ideals. Their proof used Boolean-valued models; the present proof does not use any specialized tools of mathematical logic.

We recall some basic definitions. An ideal $\mathfrak{I}$ in $\mathcal{P}(A)$ is proper if $A \notin \mathfrak{I}$. $\mathfrak{I}$ is κ-complete if the union of every subfamily of $\mathfrak{I}$ of cardinality $< \kappa$ belongs to $\mathfrak{I}$. Thus "ω_1-complete" means "σ-complete". We set $\mathfrak{I}^+ = \mathcal{P}(A)\backslash\mathfrak{I}$, i.e. $\mathfrak{I}^+$ is the family of "$\mathfrak{I}$-large" sets. $\mathfrak{I}$ is κ-saturated if every disjoint subcollection of $\mathfrak{I}^+$ has cardinality $< \kappa$. For $A' \subseteq A$, the relativization of $\mathfrak{I}$ to A' is simply the ideal $\mathfrak{I} \cap \mathcal{P}(A')$, in $\mathcal{P}(A')$.

We shall work extensively with sequences of 0's and 1's of countable ordinal length, so we introduce here some specialized notation pertaining to this. We shall reserve the letters s and t to denote such sequences which, in addition, have only finitely many zero entries. We shall use the letter σ to denote arbitrary such sequences. As usual, $\sigma \restriction \alpha$ is the restriction of σ to α. $s \subsetneq t$ means that s is a proper initial segment of t. $\bar{\sigma}$ (or $\bar{s}$) is the sequence obtained from σ by changing the first zero entry to one, and otherwise keeping σ unchanged. 1^α denotes the sequence s of length α, all of whose entries are $= 1$. si denotes the sequence whose length $= \ell(s) + 1$, where s is an initial segment and the last entry $= i$.

Definition. An ideal $\mathfrak{I}$ in $\mathcal{P}(A)$ is *impure* if there exist disjoint X, Y in $\mathfrak{I}^+$ and a 1-1 onto function $f : X \to Y$ such that for all $X' \subseteq X$, $X' \in \mathfrak{I}^+$ iff $f(X') \in \mathfrak{I}^+$. $\mathfrak{I}$ is *pure* if it is not impure.

The following theorem has been obtained by P. Chernoff and R. Solovay in their investigation of the structure of Gleason measures on non-separable spaces.

Theorem. *If $\mathfrak{I}$ is an ω_1-saturated ω_1-complete, proper ideal in $\mathcal{P}(A)$, then there is an A' in $\mathfrak{I}^+$ such that the relativization of $\mathfrak{I}$ to A' is pure.*

Their proof used the Boolean-valued models. Here we give a proof which does not use any specialized tools of mathematical logic. The same type of

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argument applies to λ-complete, λ-saturated ideals for any uncountable cardinal λ.

We assume from the start that the theorem is false and all of our lemmas and constructions rely on this assumption. Thus in particular let $\mathfrak{I}$ be an ω_1-saturated, ω_1-complete, proper ideal in $\mathcal{P}(A)$, for some set A, and let the relativization of $\mathfrak{I}$ to all subsets A' in $\mathfrak{I}^+$ be impure. We also assume that A is endowed with a well-ordering, thus without loss of generality, let A be a set of ordinal numbers.

Lemma 1. *If $A' \in \mathfrak{I}^+$, then there exists a partition $A' = A''\dot{\cup}A'''$, with A'', A''' belonging to $\mathfrak{I}^+$, and a* 1-1 *onto function $f : A'' \to A'''$ such that $f(x) < x$ for all x in A'', and for all $X \subseteq A''$, $X \in \mathfrak{I}^+$ iff $f(X) \in \mathfrak{I}^+$.*

Proof. It is easy to see that one can augment the definition of impure ideals by requiring the additional condition that $f(x) < x$ (when A is endowed with an ordering $<$).

Now let (X_i, Y_i, f_i), for i in some index set I, be a maximal collection satisfying the following conditions: $X_i \cup Y_i \subseteq A'$, $X_i \cap Y_i = \varnothing$, X_i, Y_i both belong to $\mathfrak{I}^+$; $f_i : X_i \to Y_i$ is 1-1 and onto, $f_i(x) < x$ for all x in X_i; for all $X \subseteq X_i$, $X \in \mathfrak{I}^+$ iff $f_i(X) \in \mathfrak{I}^+$; and for all i and j in I, if $i \neq j$, then $X_i \cup Y_i$ is disjoint from $X_j \cup Y_j$.

By the ω_1-saturation, the maximal collection above is at most enumerable. Now set

$$X = \bigcup_{i\in I} X_i, \quad Y = \bigcup_{i\in I} Y_i.$$

If $A'\backslash(X \cup Y)$ belongs to $\mathfrak{I}^+$, then the relativization of $\mathfrak{I}$ to this set is impure, and we readily obtain a contradiction with the maximality of the family (X_i, Y_i, f_i) above. Thus $A'\backslash(X \cup Y) \in \mathfrak{I}$. If this set is non-empty, we can assume that it is infinite. To see this, note that all singletons belong to $\mathfrak{I}$, for if a singleton $\{x\}$ belongs to $\mathfrak{I}^+$, then the relativization of $\mathfrak{I}$ to $\{x\}$ is pure, contradicting our assumption. Thus all enumerable sets belong to $\mathfrak{I}$. Now if $A'\backslash(X \cup Y)$ is finite, we can augment it by an additional infinite enumerable set from some X_i, together with the image of this set under f_i. Thus it is clearly possible to find a partition

$$A'\backslash(X \cup Y) = X'\dot{\cup}Y'$$

and a 1-1 onto function $f' : X' \to Y'$ satisfying the additional condition $f'(x) < x$ for all x in X'.

Finally let $A'' = X \cup X'$, $A''' = Y \cup Y'$, and let

$$f = \left(\bigcup_{i\in I} f_i\right) \cup f'.$$

We shall define sets $A(s) \subseteq A$ for sequences s of zeros and ones, with finitely many zeros only, and of countable ordinal length $\ell(s)$. We shall also define functions f_β, for β in ω_1. The following conditions will be satisfied.

(1) $A(\varnothing) = A$.
(2) $A(s) = A(s0)\dot{\cup}A(s1)$.
(3) If $\ell(s) = \beta$ is limit, then $A(s) = \cap_{\gamma<\beta} A(s \restriction \gamma)$.
(4) If $t \supseteq s$, then $A(t) \subseteq A(s)$.

(5) $\operatorname{dom}(f_\beta) = A(1^\beta 0)$.

(6) If $A(1^\beta) \in \mathfrak{I}$, then f_β is the identity on $A(1^\beta 0) = A(1^\beta)$, and $A(1^{\beta+1}) = \varnothing$.

(7) If $A(1^\beta) \in \mathfrak{I}^+$, then $A(1^\beta i) \in \mathfrak{I}^+$ for $i = 0, 1$, f_β is one-to-one, range $(f_\beta) = A(1^{\beta+1})$, $f_\beta(x) < x$ for all x in $A(1^\beta 0)$, and for all $X \subseteq A(1^\beta 0)$, $X \in \mathfrak{I}^+$ iff $f_\beta(X) \in \mathfrak{I}^+$.

(8) If $s \supseteq 1^\beta 0$ and $A(1^\beta 0) \in \mathfrak{I}^+$, then $A(s) = f_\beta^{-1} A(\bar{s})$.

We now give a transfinite induction definition of the sets $A(s)$ and functions f_β satisfying the conditions (1)–(8). We repeat that only sequences with at most finitely many zeros are considered. At the limit ordinal stage we proceed as follows: If α is limit and the sets $A(s)$ and the functions f_β are defined for $\ell(s) < \alpha$ and $\beta < \alpha$, we shall define $A(s)$ for sequences s, with $\ell(s) = \alpha$, in conformity with (3). The functions f_α, irrespective of whether α is limit or successor, will be defined during the passage from α to $\alpha + 1$, i.e. while we are defining the sets $A(s)$ for sequences s whose length $\ell(s) = \alpha + 1$.

Let us first consider briefly the limit stage. It is easy to see that the validity of the conditions (1)–(8) "extends one stage higher". Let us check, for example, (8). Let α, $\alpha \neq 0$, be the limit ordinal under consideration. We only need to check (8) for $\beta < \alpha$, $\ell(s) = \alpha$. By the inductive assumption we have, for γ which are $> \beta$ and $< \alpha$, $A(s \restriction \gamma) = f_\beta^{-1} A(\overline{s \restriction \gamma})$. Thus

$$A(s) = \bigcap_{\beta<\gamma<\alpha} A(s \restriction \gamma) = \bigcap_{\beta<\gamma<\alpha} f_\beta^{-1} A(\overline{s \restriction \gamma})$$
$$= f_\beta^{-1} \bigcap_{\beta<\gamma<\alpha} A(\bar{s} \restriction \gamma) = f_\beta^{-1} A(\bar{s}).$$

(By (4), the first and the last equality are certainly valid.) We omit the discussion of the ($\alpha = 0$)-case except to say that (1) then constitutes the definition of $A(\varnothing)$.

We now turn the attention to the successor stage. So suppose that $A(s)$ are defined for sequences s of length $\ell(s) \leq \alpha$, that the f_β are defined for $\beta < \alpha$, and that (1)–(8) are satisfied in the appropriately relativized form, i.e. (1) holds, (2) holds when $\ell(s) < \alpha$, (3) holds for all limit $\beta \leq \alpha$, (4) holds for all t and s of length $\leq \alpha$, (5), (6) and (7) hold for all $\beta < \alpha$, and (8) holds for all $\beta < \alpha$ and all s of length $\leq \alpha$. We will define f_α and the sets $A(s)$ for sequences s of length $\alpha + 1$. We begin with $A(1^\alpha 0)$, $A(1^{\alpha+1})$, and f_α. If $A(1^\alpha) \in \mathfrak{I}$, read (5) and (6), with α in place of β, as the relevant definitions. Now let $A(1^\alpha) \in \mathfrak{I}^+$. Since the relativization of $\mathfrak{I}$ to no subset Y of $A(1^\alpha)$ is pure, we can obtain a partition $A(1^\alpha) = A(1^\alpha 0) \dot{\cup} A(1^{\alpha+1})$ and $f_\alpha : A(1^\alpha 0) \to A(1^{\alpha+1})$, 1-1, onto, and such that $f(x) < x$ for all x in the domain of f, and for $X \subseteq A(1^\alpha 0)$, $X \in \mathfrak{I}^+$ if and only if $f(X) \in \mathfrak{I}^+$. This gives (7), with α in place of β. It remains to define $A(si)$ when $\ell(s) = \alpha$, $s \neq 1^\alpha$, and $i = 0, 1$. This we do simultaneously for both cases $A(1^\alpha) \in \mathfrak{I}$, $\mathfrak{I}^+$. Let β be minimal such that the β-th entry of s is equal to 0. Then $\beta < \alpha$. Proceeding by induction on the number of zero entries of s (of which there are only finitely many), we may assume that $A(\bar{s}i)$, for $i = 0, 1$, are already defined. We set $A(si) = f_\beta^{-1} A(\bar{s}i)$ if $A(1^\beta 0) \in \mathfrak{I}^+$, and $A(s0) = A(s)$, $A(s1) = \varnothing$ if $A(1^\beta 0) \in \mathfrak{I}$.

We now need to verify that (1)–(8) hold "one stage further". Consider (8): We need to verify the statement for sequences s of length $\alpha + 1$ and all $\beta \leq$

α. First let $\beta = \alpha$. Then $s = 1^\alpha 0$ and $A(1^\alpha 0) \in \mathfrak{J}^+$, thus clearly, by the construction, $A(s) = A(1^\alpha 0) = f_\alpha^{-1} A(1^{\alpha+1}) = f_\alpha^{-1} A(\bar{s})$.

Now let $\beta < \alpha$. Let us also write a sequence of length $\alpha + 1$ in the form si where $\ell(s) = \alpha$ and $i = 0$ or 1. We suppose that $si \supseteq 1^\beta 0$ and $A(1^\beta 0)$ belongs to $\mathfrak{J}^+$. Clearly $s \supseteq 1^\beta 0$ and thus, by the construction, $A(si) = f_\beta^{-1} A(\bar{s}i)$. But $\bar{s}i = \overline{si}$, hence the desired conclusion. Thus (8) holds "one stage further".

Next we verify (2). For $s = 1^\alpha$ the conclusion is immediate from the construction. So let $s \neq 1^\alpha$, $\ell(s) = \alpha$. Again let $\beta < \alpha$ be minimal such that the β-th entry of s is equal to 0. We consider only the case when $A(1^\beta 0) \in \mathfrak{J}^+$. By the induction hypothesis, $A(s) = f_\beta^{-1} A(\bar{s})$ and $A(\bar{s}) = A(\bar{s}0) \dot{\cup} A(\bar{s}1)$ can be assumed since we also wish to use induction on the number of zero entries of s. (The validity of the case $s = 1^\alpha$ has already been noted.) We thus obtain

$$A(s) = f_\beta^{-1} A(\bar{s}0) \dot{\cup} f_\beta^{-1} A(\bar{s}1) = A(s0) \dot{\cup} A(s1).$$

The verification of the remaining clauses should not pose any difficulty.

Lemma. *The following are equivalent:*

(i) $A(1^\alpha) \in \mathfrak{J}^+$;
(ii) *for some* s *with* $\ell(s) = \alpha$, $A(s) \in \mathfrak{J}^+$;
(iii) *for all* s *of length* α, $A(s) \in \mathfrak{J}^+$.

Proof. The result clearly follows if we show that (ii) implies (i) and (i) implies (iii). So first assume (ii). Then we may assume that s has at least one zero entry since in the contrary case $s = 1^\alpha$ and (i) follows. So let $\beta_0 < \beta_1 < \cdots < \beta_k$ be the (finite) list of ordinals β such that the β-th entry of s is equal to 0. Then by (8), (7), and (4),

$$f_{\beta_\kappa} f_{\beta_{\kappa-1}} \ldots f_{\beta_0} A(s) \subseteq A(1^\alpha)$$

and the set on the left-hand side of this inclusion belongs to $\mathfrak{J}^+$ since $A(s) \in \mathfrak{J}^+$. Hence $A(1^\alpha) \in \mathfrak{J}^+$.

Next suppose (i). We prove (iii) by induction on the number of zero entries of s. We also (inductively) assume (iii) to hold for all $\beta < \alpha$. Again, if s has no zero entries, then $s = 1^\alpha$ and we are done. So let s have at least one zero entry and let β be minimal such that the β-th entry of s is equal to 0. Then $A(s) = f_\beta^{-1} A(\bar{s})$, $A(\bar{s}) \subseteq A(1^{\beta+1})$, and $A(\bar{s}) \in \mathfrak{J}^+$ since $\bar{s}$ has fewer zero entries than s. Thus $A(s) \in \mathfrak{J}^+$ by (7).

Lemma. *There is a sequence* σ, *of limit ordinal length, such that* σ *has infinitely many zero entries, for every proper initial segment* s *of* σ, s *has only finitely many zero entries, and* $A(s) \in \mathfrak{J}^+$, *and the intersection of these* $A(s)$ *is nonempty.*

Proof. There must be some $\alpha \in \omega_1$ such that $A(1^\alpha) \in \mathfrak{J}$. For in the contrary case the sets $A(1^\alpha)$ are decreasing sets from $\mathfrak{J}^+$ such that the $A(1^\alpha) \setminus A(1^{\alpha+1})$ also belong to $\mathfrak{J}^+$, contradicting the ω_1-saturation of $\mathfrak{J}$. So let α be minimal such that $A(1^\alpha) \in \mathfrak{J}$. Clearly α is a limit ordinal.

We claim that there is some sequence σ, of zeros and ones, such that $\ell(\sigma)$ is limit, $\ell(\sigma) \leq \alpha$, σ has infinitely many zeros, each proper initial segment of σ has only finitely many zeros, and

$$\bigcap_{\beta < \ell(\sigma)} A(\sigma \restriction \beta) \neq 0.$$

Applying the preceding lemma with $\beta < \alpha$ in place of α, we conclude that $A(\sigma \upharpoonright \beta) \in \mathfrak{I}^+$ for all $\beta < \ell(\sigma)$.

We will establish the claim by contradiction. So suppose that there is no σ as claimed. We will now show that for all $\beta \leq \alpha$, the union of the $A(s)$, corresponding to the sequences s of length β, covers A. We proceed by induction on β. The case $\beta = 0$ follows from (1), and the successor stage follows from (2). So let β be limit and x belong to A. By the induction hypothesis, for every $\gamma < \beta$, x belongs to some $A(s_\gamma)$ with $\ell(s_\gamma) = \gamma$. Clearly each of these s_γ has only finitely many zero entries. Set $\sigma = \bigcup_{\gamma<\beta} s_\gamma$. Then σ is a sequence of length β by (4). If σ had infinitely many zero entries, σ would fulfill the requirements of the claim. Thus σ has only finitely many zero entries and therefore $A(\sigma)$ is defined and $x \in A(\sigma)$. Thus for every $\beta \leq \alpha$, the union of the sets $A(s)$, where $\ell(s) = \beta$, covers A. But there are only enumerably many sequences s, of any fixed length, with at most finitely many zero entries. It follows that some $A(s)$, with $\ell(s) = \alpha$, belongs to $\mathfrak{I}^+$. Thus by the preceding lemma, $A(1^\alpha) \in \mathfrak{I}^+$. This contradiction establishes the claim, and thus the lemma.

Proof of the theorem. Let σ be as in the preceding lemma and let $\langle \beta_i : i \in \omega \rangle$ list, in increasing order, all β such that the β-th entry of s is equal to zero. Let x be any element of

$$\bigcap_{s \subsetneq \sigma} A(s).$$

Then $\sigma \supseteq 1^{\beta_0}0$ and $A(1^{\beta_0}0) \in \mathfrak{I}^+$, hence for all proper initial segments s of σ such that $\ell(s) \geq \beta_0 + 1$, we get $A(s) = f_{\beta_0}^{-1} A(\bar{s})$ and thus $f_{\beta_0}(x)$ belongs to

$$\bigcap_{t \subsetneq \bar{\sigma}} A(t).$$

We also have $f_{\beta_0}(x) < x$. From the above equation $A(s) = f_{\beta_0}^{-1} A(\bar{s})$ we also conclude that $A(t)$ belongs to $\mathfrak{I}^+$ for all proper initial segments t of $\bar{\sigma}$. Thus we can repeat the preceding argument with $\bar{\sigma}$ in place of σ and obtain

$$f_{\beta_1} f_{\beta_0}(x) < f_{\beta_0}(x).$$

Continuing this process gives an infinite decreasing sequence of ordinals.

Thus the theorem is proved.

Acknowledgment

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