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EQUIVARIANT, ALMOST HOMEOMORPHIC MAPS BETWEEN S^1 AND S^2

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ABSTRACT. Let Π be a Fuchsian group isomorphic to a non-trivial, closed surface group, and let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold admitting an isomorphism $\rho\colon \Pi \to \Gamma$. Under certain assumptions, Cannon-Thurston and Minsky showed that there exists a ρ-equivariant, surjective, continuous map $f\colon S^1_\infty \to S^2_\infty$. In this paper, we prove that there exist zero-measure sets Λ^1 in S^1_∞ and Λ^2 in S^2_∞ such that the restriction $f|_{S^1_\infty - \Lambda^1}\colon S^1_\infty - \Lambda^1 \to S^2_\infty - \Lambda^2$ is a homeomorphism.

For any countable sets C_1 in S^1 and C_2 in S^2, $S^1 - C_1$ is not homeomorphic to $S^2 - C_2$. In fact, $S^2 - C_2$ contains infinitely many, mutually disjoint, simple loops, but $S^1 - C_1$ does not. Here, we consider the problem whether there exist zero-measure sets N_1 in S^1 and N_2 in S^2 such that $S^1 - N_1$ is homeomorphic to $S^2 - N_2$.

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold homotopy-equivalent to a closed, connected, orientable, hyperbolic surface $\Sigma_g = \mathbb{H}^2/\Pi$ of genus $g > 1$. A homotopy-equivalent map from Σ_g to M induces the isomorphism $\rho\colon \Pi \to \Gamma$. It is well known that the isometric action of Π on $\mathbb{H}^2$ (resp. Γ on $\mathbb{H}^3$) is extended continuously to that on the circle S^1_∞ (resp. the sphere S^2_∞) at infinity. If M contains no geometrically finite ends and the injectivity radius $\mathrm{inj}(M) = \inf\{\mathrm{inj}_M(x)\,;\, x \in M\} > 0$, then by Minsky [6] (see also Cannon-Thurston [2]), there exists a ρ-equivariant, continuous map $f\colon S^1_\infty \to S^2_\infty$. Here, f *ρ-equivariant* means that, for any $\gamma \in \Pi$ and any $x \in S^1_\infty$, f satisfies $f(\gamma x) = \rho(\gamma) f(x)$. Consider the subset Λ^2 of S^2_∞ consisting of all points $x \in S^2_\infty$ such that $f^{-1}(x)$ has at least two elements, and set $\Lambda^1 = f^{-1}(\Lambda^2)$.

In this paper, we prove the following theorem.

Theorem. *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold homotopy-equivalent to a closed, connected, orientable, hyperbolic surface $\Sigma_g = \mathbb{H}^2/\Pi$. Suppose that $\mathrm{inj}(M) > 0$ and M has no geometrically finite ends. Then, for the isomorphism $\rho\colon \Pi \to \Gamma$, a continuous map $f\colon S^1_\infty \to S^2_\infty$, and the subsets $\Lambda^1 \subset S^1_\infty$, $\Lambda^2 \subset S^2_\infty$ given as above, both the 1-dimensional Lebesgue measure of Λ^1 in S^1_∞ and*

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the 2-dimensional Lebesgue measure of Λ^2 *in* S^2_∞ *are zero. Furthermore, the restriction* $f|_{S^1_\infty - \Lambda^1}: S^1_\infty - \Lambda^1 \to S^2_\infty - \Lambda^2$ *is a* ρ*-equivariant homeomorphism.*

This theorem is a result which belongs not only to general topology or Lebesgue measure theory but also to hyperbolic geometry.

Let $M' = \mathbb{H}^3/\Gamma'$ be another hyperbolic 3-manifold satisfying the same conditions as M does, and let $\rho': \Pi \to \Gamma'$ be an isomorphism. For a ρ'-equivariant, continuous map $f': S^1_\infty \to S^2_\infty$, the $\rho' \circ \rho^{-1}$-equivariant, continuous map $f' \circ (f|_{S^1_\infty - \Lambda^1})^{-1}: S^2_\infty - \Lambda^2 \to S^2_\infty$ is useful to compare Γ with Γ' directly. For example, in Soma [7], by using this $\rho' \circ \rho^{-1}$-equivariant map, it is shown that, if the fundamental classes of M and M' in the third bounded cohomology $H^3_b(\Sigma_g, \mathbb{R})$ are sufficiently close to each other with respect to the pseudo-norm, then M and M' have the same ending invariants, and hence M is isometric to M' by Minsky's Ending Lamination Theorem [6].

1. Cannon-Thurston-Minsky construction

We refer to Thurston [10] for the fundamental notation and definitions on hyperbolic geometry.

Let Σ_g be a closed, connected, orientable, hyperbolic surface of genus $g > 1$. A closed subset of Σ_g is a *geodesic lamination* if it consists of mutually disjoint, simple geodesics (called *leaves* of the lamination). A *measured lamination* μ on Σ_g is a geodesic lamination with invariant transverse measure. The underlying geodesic lamination of μ is called the *support* of μ and denoted by $|\mu|$; see [10, Chapter 8] and [3] for more information on laminations. A *measured foliation* λ on Σ_g is a topological foliation on Σ_g with saddle singularities, equipped with transverse invariant measure; we refer to [11] and [4] for details on measured laminations. It is well known that there exists the natural one-to-one correspondence between the set of measured laminations on Σ_g and that of equivalent classes of measured foliations on Σ_g; for example see Levitt [5].

For $n = 2, 3$, we denote by $\mathbb{B}^n$ the unit n-ball model for the union $\mathbb{H}^n \cup S^{n-1}_\infty$ of the hyperbolic n-space and the $(n-1)$-sphere at infinity. For the Fuchsian group Π with $\Sigma_g = \mathbb{H}^2/\Pi$, the action of Π on $\mathbb{B}^2$ is naturally extended to the isometric action on $\mathbb{H}^3$ and the conformal action on S^2_∞. Let H_+, H_- be the closures of components of $S^2_\infty - S^1_\infty = \partial\mathbb{B}^3 - \partial\mathbb{B}^2$ in S^2_∞, and let $p_+: \operatorname{int} H_+ \to \operatorname{int} H_+/\Pi = \Sigma_g$, $p_-: \operatorname{int} H_- \to \operatorname{int} H_-/\Pi = \Sigma_g$ and $q: \mathbb{H}^2 \to \mathbb{H}^2/\Pi = \Sigma_g$ be the universal coverings. For two measured foliations λ_+, λ_- on Σ_g, we set $\tilde{\lambda}_+ = p_+^{-1}(\lambda_+) \subset \operatorname{int} H_+$, $\tilde{\lambda}_- = p_-^{-1}(\lambda_-) \subset \operatorname{int} H_-$ and $\hat{\lambda}_+ = q^{-1}(\lambda_+)$, $\hat{\lambda}_- = q^{-1}(\lambda_-) \subset \mathbb{H}^2$. Consider the projection $\pi: \mathbb{B}^3 \to \mathbb{B}^2$ defined so that, for any $x \in \mathbb{B}^2 \subset \mathbb{B}^3$, $\pi(x) = x$, and for any $y \in \mathbb{B}^3 - \mathbb{B}^2$, $\pi(y)$ is the intersection point of $\operatorname{int}\mathbb{B}^2 = \mathbb{H}^2$ with the geodesic line l in $\mathbb{H}^3$ meeting $\mathbb{H}^2$ orthogonally and satisfying $\operatorname{cl}(l) \ni y$, where $\operatorname{cl}(l)$ is the closure of l in $\mathbb{B}^3$. Then, we have $\pi(\tilde{\lambda}_+) = \hat{\lambda}_+$ and $\pi(\tilde{\lambda}_-) = \hat{\lambda}_-$.

Consider a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ with $\operatorname{inj}(M) > 0$ and admitting an isomorphism $\rho: \Pi \to \Gamma \subset \operatorname{Isom}^+(\mathbb{H}^3)$. According to Minsky [6, §7], if M contains no geometrically finite ends, then there exist two measured foliations λ_+, λ_- on Σ_g and a ρ-equivariant, continuous map $F: \mathbb{B}^3 \to \mathbb{B}^3$ such that, for any leaves l_+ of $\tilde{\lambda}_+$ and l_- of $\tilde{\lambda}_-$, $\operatorname{cl}(l_+) \cap \operatorname{cl}(l_-) = \varnothing$ and such that, for any points $x, y \in S^2_\infty$ with $x \neq y$, $F(x) = F(y)$ if and only if there exists a leaf l of either $\tilde{\lambda}_+$ or $\tilde{\lambda}_-$ with $\{x, y\} \subset \mathrm{cl}(l)$.

The supports $|\mu_+|, |\mu_-|$ of the measured laminations μ_+, μ_- on Σ_g corresponding to λ_+ and λ_- are called the *ending laminations* for M. Since there exists the natural one-to-one correspondence between the set $\mathscr{G}$ of leaves in $\tilde{\lambda}_+ \cup \tilde{\lambda}_-$ not homeomorphic to the open interval and the set of connected components of $(\mathrm{int}\, H_+ - p_+^{-1}(|\mu_+|)) \cup (\mathrm{int}\, H_- - p_-^{-1}(|\mu_-|))$, $\mathscr{G}$ is a countable set. By [10, Proposition 9.3.8], each component of $\Sigma_g - |\mu_+|$ and $\Sigma_g - |\mu_-|$ is a finite-sided polygon with ideal vertices. It follows that, for each leaf l in $\mathscr{G}$, $\mathrm{cl}(l) \cap S^1_\infty$ consists of finitely many points. We say that

$$A_\Gamma = \{x \in S^1_\infty \,;\, x \in \mathrm{cl}(l) \text{ for some } l \in \mathscr{G}\}$$

is the *countable, exceptional set* for Γ. Since $F(S^1_\infty)$ is a Γ-invariant, closed subset of S^2_∞ and since the limit set of Γ is S^2_∞, $F(S^1_\infty)$ coincides with S^2_∞. Thus, the restriction

$$f = F|_{S^1_\infty} : S^1_\infty \to S^2_\infty$$

is a ρ-equivariant, surjective, continuous map. We set

$$\Lambda^1_\pm = \{x \in S^1_\infty \,;\, x \in \mathrm{cl}(l) \text{ for some leaf } l \text{ of } \tilde{\lambda}_\pm\},$$

and $\Lambda^1 = \Lambda^1_+ \cup \Lambda^1_-$, $\Lambda^2_\pm = f(\Lambda^1_\pm)$, $\Lambda^2 = f(\Lambda^1) = \Lambda^2_+ \cup \Lambda^2_-$.

The existence of such a map F was first shown by Cannon and Thurston [2] in special cases and by Minsky [6] for any M satisfying the above conditions.

2. Proof of Theorem

Let μ_1, μ_2 be respectively the 1-dimensional and 2-dimensional Lebesque measures on S^1_∞, S^2_∞ with respect to the fixed euclidean metrics on $\mathbb{B}^2$ and $\mathbb{B}^3$. The following lemma is the essential part of Theorem.

Lemma 1. $\mu_2(\Lambda^2) = \mu_2(\Lambda^2_+) + \mu_2(\Lambda^2_-) = 0$.

Proof. We will show that $\mu_2(\Lambda^2_+) = 0$. It is proved similarly that $\mu_2(\Lambda^2_-) = 0$. Let α_0 be a (short) geodesic segment in $\mathbb{H}^2$ meeting leaves of $\widehat{\lambda}_+$ transversely. If necessary replacing α_0 by a sufficiently shorter subsegment, we may assume:

(2.1) For each leaf l of $\widehat{\lambda}_+$ meeting α_0 non-trivially, $l \cap \alpha_0$ consists of a single point.

Consider the set X_0 of all points $x \in \mathbb{B}^2$ such that $x \in \mathrm{cl}(l)$ for some leaf l of $\widehat{\lambda}_+$ with $l \cap \alpha_0 \neq \varnothing$. Note that X_0 is a closed (and hence compact) subset of $\mathbb{B}^2$; see Figure 1 on the next page.

Set $\Pi = \{\gamma_0, \gamma_1, \gamma_2, \dots\}$ so that $\gamma_0 = 1$, and let $X_n = \gamma_n X_0$, $\alpha_n = \gamma_0 \alpha_0$ for all $n \in \mathbb{N}$. By [10, Proposition 9.3.8], for each leaf l of $\hat{\lambda}_+$, the image $q(l)$ is dense in Σ_g. This shows that there exists $\gamma_n \in \Pi$ such that $\gamma_n^{-1} l \cap \alpha_0 \neq \varnothing$ or equivalently $l \subset X_n$. Thus, we have $\Lambda^1_+ \cup \mathbb{H}^2 = \bigcup_{n=0}^\infty X_n$. For each $n \in \{0\} \cup \mathbb{N}$, $Y_n = \pi^{-1}(X_n) \cap H_+$ and $\beta_n = \pi^{-1}(\alpha_n) \cap H_+$ are homeomorphic respectively to X_n and α_n.

Since $F(Y_n)$ is a compact (and hence closed) subset of S^2_∞, $\Lambda^2_+ = \bigcup_{n=0}^\infty F(Y_n)$ is a Γ-invariant Borel set. Since the limit set of Γ is the whole sphere S^2_∞, by Sullivan [8] (see also Canary [1, §9]), there are no positive non-constant superharmonic functions on M. Then, by Sullivan [9], the solid angle

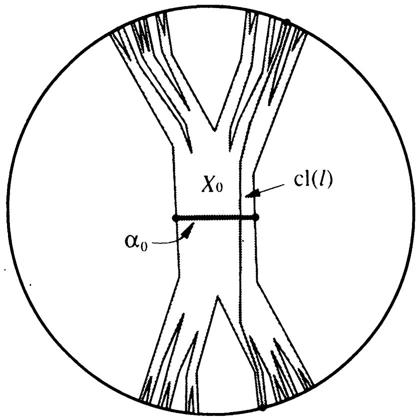

FIGURE 1

$\sum_{\gamma\in\Gamma}\exp(-2\mathrm{dist}(x_0, \gamma x_0))$ of Γ, $x_0 \in \mathbb{H}^3$, is infinite, and the action of Γ on S^2_∞ is ergodic. This implies that either $\mu_2(\Lambda^2_+) = 0$ or $\mu_2(S^2_\infty - \Lambda^2_+) = 0$. Here, we suppose that $\mu_2(S^2_\infty - \Lambda^2_+) = 0$ and induce a contradiction. If $\mu_2(F(Y_n)) = 0$ for some $n \in \{0\}\cup\mathbb{N}$, then for each $m \in \{0\}\cup\mathbb{N}$, $\mu_2(F(Y_m)) = \mu_2(\rho(\gamma_m\cdot\gamma_n^{-1})F(Y_n)) = 0$ and hence $\mu_2(\Lambda^2_+) = 0$. Thus, $\mu_2(S^2_\infty - \Lambda^2_+) = 0$ implies that, for each $n \in \{0\}\cup\mathbb{N}$, $\mu_2(F(Y_n)) > 0$. Since, by (2.1), the restriction $F_n = F|_{\beta_n}: \beta_n \to F(\beta_n) = F(Y_n)$ is injective and since F_n is a closed map, F_n is a homeomorphism. We will define the Γ-invariant map $\eta: S^2_\infty \times S^2_\infty \to \mathbb{R}$ as follows. Set $\eta(x, y) = 0$ if $(x, y) \in S^2_\infty \times S^2_\infty - \Lambda^2_+\times\Lambda^2_+$. For $(x, y) \in \Lambda^2_+\times\Lambda^2_+$, we set $\eta(x, y) = \mathrm{dist}_{\mathbb{H}^2}(l_x, l_y)$, where l_x, l_y are the leaves of $\hat{\lambda}_+$ with $l_x \subset \pi(F^{-1}(x))$, $l_y \subset \pi(F^{-1}(y))$. Obviously, η is Γ-invariant. For any $m, n \in \{0\}\cup\mathbb{N}$ (possibly $m = n$), we will show that the restriction $\eta|_{F(Y_m)\times F(Y_n)}$ is a measurable function. By the continuities for $F_m^{-1}: F(Y_m) \to \beta_m$ and $F_n^{-1}: F(Y_n) \to \beta_n$, it is proved that

$$\eta(x, y) = \mathrm{dist}_{\mathbb{H}^2}(\pi(l_{F_m^{-1}(x)}), \pi(l_{F_n^{-1}(y)}))$$

is continuous in $R_{m,n} = F(Y_m)\times F(Y_n) - F(A_\Gamma)\times F(Y_n)\cup F(Y_m)\times F(A_\Gamma)$, where $l_{F_m^{-1}(x)}$, $l_{F_n^{-1}(y)}$ are the leaves of $\tilde{\lambda}_+$ containing $F_m^{-1}(x)$ and $F_n^{-1}(y)$ respectively. In fact, for any $(x, y) \in R_{m,n}$, there exist mutually disjoint 2-disks D_1, D_2 (resp. D_3, D_4) in $\mathbb{B}^2$ which are closed neighborhoods of the end points of $\mathrm{cl}(l_x)$, where $l_x = \pi(l_{F_m^{-1}(x)})$ (resp. of $\mathrm{cl}(l_y)$, where $l_y = \pi(l_{F_n^{-1}(y)})$) with $\alpha_m \cap (D_1\cup D_2) = \emptyset$ (resp. $\alpha_n\cap(D_3\cup D_4) = \emptyset$) and such that, in the case of $x \neq y$, $\mathrm{dist}_{\mathbb{H}^2}((l_x\cup D_1\cup D_2)\cap\mathbb{H}^2, (D_3\cup D_4)\cap\mathbb{H}^2) \geq 2\eta(x, y)$ and $\mathrm{dist}_{\mathbb{H}^2}((D_1\cup D_2)\cap\mathbb{H}^2, (l_y\cup D_3\cup D_4)\cap\mathbb{H}^2) \geq 2\eta(x, y)$. For any point x' in a sufficiently small neighborhood U_x of x in $F(Y_m)$, either $l_{x'}$ is homeomorphic to the closed interval or each branched point of $l_{x'}$ is contained in $D_1\cup D_2$. Then, for any $\varepsilon > 0$, one can take U_x so small that, for any $x' \in U_x$, $l_{x'} - D_1\cup D_2$ is contained in the $\varepsilon/2$-neighborhood of $l_x - D_1\cup D_2$ in $\mathbb{H}^2$ and vice versa. We have the similar situation also in a small neighborhood of y in $F(Y_n)$. This shows that η is continuous in $R_{m,n}$. See Figure 2 for a typical example of the discontinuity for η in $F(A_\Gamma)\times F(Y_n)\cup F(Y_m)\times F(A_\Gamma)$. In Figure 2, though $\{x_n\}\subset S^2_\infty$ converges to $x \in F(A_\Gamma)$ and $\{\eta(x_n, y)\}$ converges to s, in general, $\eta(x, y) = t$ does not coincide with s.

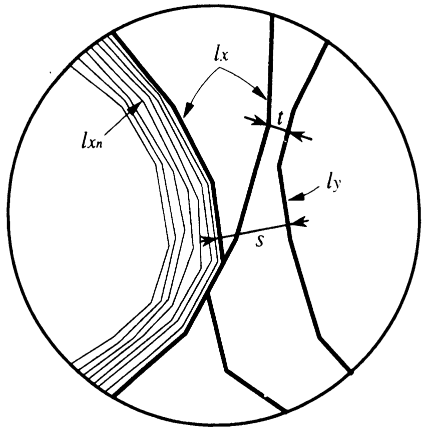

FIGURE 2

Since A_Γ is a countable set, for the product measure $\mu_2^2 = \mu_2 \times \mu_2$ on $S^2_\infty \times S^2_\infty$, $\mu_2^2(F(A_\Gamma) \times F(Y_n) \cup F(Y_m) \times F(A_\Gamma)) = 0$. This proves that $\eta|_{F(Y_m)\times F(Y_n)}$ is a measurable function. Since $\bigcup_{m,n=0}^{\infty} F(Y_m) \times F(Y_n) = \Lambda^2_+ \times \Lambda^2_+$ has full μ_2^2-measure in $S^2_\infty \times S^2_\infty$, η is a measurable function on $S^2_\infty \times S^2_\infty$. Since, noted as above, the solid angle of Γ is infinite, by Sullivan [9, Theorem II], Γ acts on $S^2_\infty \times S^2_\infty$ ergodically. Thus, there exists a subset N of $S^2_\infty \times S^2_\infty$ with $\mu_2^2(N) = 0$ and such that $\lambda|_{S^2_\infty \times S^2_\infty - N}$ is a constant R. Note that there exists X_n such that $\mathrm{dist}_{\mathbb{H}^2}(X_0, X_n) \geq R + 1$. Since $\mu_2^2(F(Y_0) \times F(Y_n)) > 0$, there exists $(y_0, y_n) \in F(Y_0) \times F(Y_n) - N$ such that $\eta(y_0, y_n) \geq R + 1$, a contradiction. Thus, we have $\mu_2(\Lambda^2_+) = 0$. This completes the proof. □

Here, it is worthwhile presenting an outline of the alternate proof of Lemma 1 given by the referee, which uses a little more information about the geometric model discussed in [2] and [6]. In fact, the referee proved that Λ^2 is in the complement of the conical limit set of Γ, and that set has zero-measure by Sullivan [9, p. 483, Corollary]. One can see this by considering the model metric σ on $\Sigma_g \times \mathbb{R}$ given in [2] and [6] such that the universal cover $(\mathbb{H}^2 \times \mathbb{R}, \tilde{\sigma})$ is ρ-equivalently quasi-isometric to the hyperbolic space $\mathbb{H}^3$. In $(\mathbb{H}^2 \times \mathbb{R}, \tilde{\sigma})$, there are hyperplanes $l \times \mathbb{R}$ where l is a leaf of either $\hat{\lambda}_+$ or $\hat{\lambda}_-$ which are totally geodesic, and map to quasi-geodesic planes in $\mathbb{H}^3$. The set Λ^2 is obtained as the images of the end points of $l \times \{0\}$ for all such leaves l. However, the ray $\{x\} \times [0, \infty)$ for $x \in l \subset \hat{\lambda}_+$ (or $\{x\} \times (-\infty, 0]$ for $x \in l \subset \hat{\lambda}_-$) also has the image terminating at the same point. This image g is quasi-geodesic in $\mathbb{H}^3$, and $p(g)$ is within bounded distance of a geodesic leaving every compact set in $M = \mathbb{H}^3/\Gamma$, where $p\colon \mathbb{H}^3 \to M$ is the universal covering. Thus, the image point is non-conical.

Though the following lemma is probably well known or a folklore, the author does not know suitable references. For completeness, we will present the proof similar to that of Lemma 1.

Lemma 2. $\mu_1(\Lambda^1) = \mu_1(\Lambda^1_+) + \mu_1(\Lambda^1_-) = 0$.

Proof. Since Λ^1_+ is a Π-invariant, measurable set and since, by [9], Π acts on S^1_∞ ergodically, either $\mu_1(\Lambda^1_+) = 0$ or $\mu_1(S^1_\infty - \Lambda^1_+) = 0$. Here, we suppose that $\mu_1(S^1_\infty - \Lambda^1_+) = 0$ and define the Π-invariant map $\xi: S^1_\infty \times S^1_\infty \to \mathbb{R}$ as follows. Set $\xi(x, y) = 0$ if $(x, y) \in S^1_\infty \times S^1_\infty - \Lambda^1_+ \times \Lambda^1_+$. Otherwise, $\xi(x, y) = \mathrm{dist}_{\mathbb{H}^2}(l_x, l_y)$, where l_x, l_y are the leaves of $\hat{\lambda}_+$ with $\mathrm{cl}(l_x) \ni x$, $\mathrm{cl}(l_y) \ni y$. The argument as in Lemma 1 shows that ξ is a measurable function. By the Hopf-Tsuji Theorem (see [9]), Π acts on $S^1_\infty \times S^1_\infty$ ergodically. It follows that ξ is constant almost everywhere, a contradiction. Thus, we have $\mu_1(\Lambda^1_+) = 0$ and similarly $\mu_1(\Lambda^1_-) = 0$. □

Proof of Theorem. It remains to prove that $f|_{S^1_\infty - \Lambda^1}: S^1_\infty - \Lambda^1 \to S^2_\infty - \Lambda^2$ is a homeomorphism. Since the restriction map is continuous and bijective, it suffices to show that it is a closed map. By the definition of relative topology, for any closed set C in $S^1_\infty - \Lambda^1$, there exists a closed (and hence compact) set C' in S^1_∞ with $C = (S^1_\infty - \Lambda^1) \cap C'$. Then, $f(C) = f((S^1_\infty - \Lambda^1) \cap C') \subset f(S^1_\infty - \Lambda^1) \cap f(C') = (S^2_\infty - \Lambda^2) \cap f(C')$. For any $y \in (S^2_\infty - \Lambda^2) \cap f(C')$, there exists $x \in C'$ with $f(x) = y$. Since $x \notin f^{-1}(\Lambda^2) = \Lambda^1$, x is contained in $C' \cap (S^1_\infty - \Lambda^1) = C$. This shows that $y \in f(C)$, or equivalently $f(C) = (S^2_\infty - \Lambda^2) \cap f(C')$. Since $f(C')$ is a compact (an hence closed) subset of S^2_∞, $f(C)$ is a closed subset of $S^2_\infty - \Lambda^2$. This completes the proof. □

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References

1. R. Canary, *Ends of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **6** (1993), 1–35.
2. J. Cannon and W. Thurston, *Group invariant Peano curves*, preprint.
3. A. Casson and S. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Math. Soc. Stud. Texts, vol. 9, Cambridge Univ. Press, Cambridge and New York, 1988.
4. A. Fathi, F. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque no. 66–67, Soc. Math. France, Paris, 1979.
5. G. Levitt, *Foliations and laminations on hyperbolic surfaces*, Topology **22** (1983), 119–135.
6. Y. Minsky, *On rigidity, limit sets and end invariants of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **7** (1994), 539–558.
7. T. Soma, *Bounded cohomology of closed surfaces*, preprint.
8. D. Sullivan, *Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension two*, Geometry Symposium (Utrecht 1980), Lecture Notes in Math., vol. 894, Springer-Verlag, Berlin and New York, 1981, pp. 127–144.
9. ——, *On the ergodic theory at infinity of an arbitrary discrete groups of hyperbolic motions*, Riemann Surfaces and Related Topics: Proceedings of the 1978 Stony Brook Conference, Ann. of Math. Stud., no. 97, Princeton Univ. Press, Princeton, NJ, 1981, pp. 465–496.
10. W. Thurston, *The geometry and topology of 3-manifolds*, Lecture notes, Princeton Univ., 1978.
11. ——, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431.

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