EQUIVARIANT, ALMOST HOMEOMORPHIC MAPS BETWEEN $S^1$ AND $S^2$

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Abstract. Let $\Pi$ be a Fuchsian group isomorphic to a non-trivial, closed surface group, and let $M = \mathbb{H}^3/\Pi$ be a hyperbolic 3-manifold admitting an isomorphism $\rho: \Pi \to \Gamma$. Under certain assumptions, Cannon-Thurston and Minsky showed that there exists a $\rho$-equivariant, surjective, continuous map $f: S^1_\infty \to S^2_\infty$. In this paper, we prove that there exist zero-measure sets $\Lambda^1$ in $S^1_\infty$ and $\Lambda^2$ in $S^2_\infty$ such that the restriction $f|_{S^1_\infty - \Lambda^1}: S^1_\infty - \Lambda^1 \to S^2_\infty - \Lambda^2$ is a homeomorphism.

For any countable sets $C_1$ in $S^1$ and $C_2$ in $S^2$, $S^1 - C_1$ is not homeomorphic to $S^2 - C_2$. In fact, $S^2 - C_2$ contains infinitely many, mutually disjoint, simple loops, but $S^1 - C_1$ does not. Here, we consider the problem whether there exist zero-measure sets $N_1$ in $S^1$ and $N_2$ in $S^2$ such that $S^1 - N_1$ is homeomorphic to $S^2 - N_2$.

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold homotopy-equivalent to a closed, connected, orientable, hyperbolic surface $\Sigma_g = \mathbb{H}^2/\Gamma$ of genus $g > 1$. A homotopy-equivalent map from $\Sigma_g$ to $M$ induces the isomorphism $\rho: \Pi \to \Gamma$. It is well known that the isometric action of $\Pi$ on $\mathbb{H}^2$ (resp. $\Gamma$ on $\mathbb{H}^3$) is extended continuously to that on the circle $S^1_\infty$ (resp. the sphere $S^2_\infty$) at infinity. If $M$ contains no geometrically finite ends and the injectivity radius $\text{inj}(M) = \inf\{\text{inj}_M(x); x \in M\} > 0$, then by Minsky [6] (see also Cannon-Thurston [2]), there exists a $\rho$-equivariant, continuous map $f: S^1_\infty \to S^2_\infty$. Here, $f$ $\rho$-equivariant means that, for any $\gamma \in \Pi$ and any $x \in S^1_\infty$, $f$ satisfies $f(\gamma x) = \rho(\gamma)f(x)$. Consider the subset $\Lambda^2$ of $S^2_\infty$ consisting of all points $x \in S^2_\infty$ such that $f^{-1}(x)$ has at least two elements, and set $\Lambda^1 = f^{-1}(\Lambda^2)$.

In this paper, we prove the following theorem.

Theorem. Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold homotopy-equivalent to a closed, connected, orientable, hyperbolic surface $\Sigma_g = \mathbb{H}^2/\Gamma$. Suppose that $\text{inj}(M) > 0$ and $M$ has no geometrically finite ends. Then, for the isomorphism $\rho: \Pi \to \Gamma$, a continuous map $f: S^1_\infty \to S^2_\infty$, and the subsets $\Lambda^1 \subset S^1_\infty$, $\Lambda^2 \subset S^2_\infty$ given as above, both the 1-dimensional Lebesgue measure of $\Lambda^1$ in $S^1_\infty$ and

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the 2-dimensional Lebesgue measure of $A^2$ in $S^2_{\infty}$ are zero. Furthermore, the restriction $f|_{S^1_{\infty} - A^1} : S^1_{\infty} - A^1 \rightarrow S^2_{\infty} - A^2$ is a $\rho$-equivariant homeomorphism.

This theorem is a result which belongs not only to general topology or Lebesgue measure theory but also to hyperbolic geometry.

Let $M' = \mathbb{H}^3/\Gamma'$ be another hyperbolic 3-manifold satisfying the same conditions as $M$ does, and let $\rho' : \Pi \rightarrow \Gamma'$ be an isomorphism. For a $\rho'$-equivariant, continuous map $f' : S^1_{\infty} \rightarrow S^2_{\infty}$, the $\rho' \circ \rho^{-1}$-equivariant, continuous map $f' \circ (f|_{S^1_{\infty} - A^1})^{-1} : S^2_{\infty} - A^2 \rightarrow S^2_{\infty}$ is useful to compare $\Gamma$ with $\Gamma'$ directly. For example, in Soma [7], by using this $\rho' \circ \rho^{-1}$-equivariant map, it is shown that, if the fundamental classes of $M$ and $M'$ in the third bounded cohomology $H^3_b(\Sigma_g, \mathbb{R})$ are sufficiently close to each other with respect to the pseudo-norm, then $M$ and $M'$ have the same ending invariants, and hence $M$ is isometric to $M'$ by Minsky's Ending Lamination Theorem [6].

1. CANNON-THURSTON-MINSKY CONSTRUCTION

We refer to Thurston [10] for the fundamental notation and definitions on hyperbolic geometry.

Let $\Sigma_g$ be a closed, connected, orientable, hyperbolic surface of genus $g > 1$. A closed subset of $\Sigma_g$ is a geodesic lamination if it consists of mutually disjoint, simple geodesics (called leaves of the lamination). A measured lamination $\mu$ on $\Sigma_g$ is a geodesic lamination with invariant transverse measure. The underlying geodesic lamination of $\mu$ is called the support of $\mu$ and denoted by $|\mu|$; see [10, Chapter 8] and [3] for more information on laminations. A measured foliation $\lambda$ on $\Sigma_g$ is a topological foliation on $\Sigma_g$ with saddle singularities, equipped with transverse invariant measure; we refer to [11] and [4] for details on measured laminations. It is well known that there exists the natural one-to-one correspondence between the set of measured laminations on $\Sigma_g$ and that of equivalent classes of measured foliations on $\Sigma_g$; for example see Levitt [5].

For $n = 2, 3$, we denote by $B^n$ the unit $n$-ball model for the union $\mathbb{H}^n \cup S^n_{\infty}$ of the hyperbolic $n$-space and the $(n - 1)$-sphere at infinity. For the Fuchsian group $\Pi$ with $\Sigma_g = \mathbb{H}^2/\Pi$, the action of $\Pi$ on $B^2$ is naturally extended to the isometric action on $H^3$ and the conformal action on $S^2_{\infty}$. Let $H_+, H_-$ be the closures of components of $S^2_{\infty} - S^1_{\infty} = \partial B^3 - \partial B^2$ in $S^2_{\infty}$, and let $p_+ : \text{int} H_+ \rightarrow \text{int} H_+ / \Pi = \Sigma_g$, $p_- : \text{int} H_- \rightarrow \text{int} H_- / \Pi = \Sigma_g$ and $q : \mathbb{H}^2 \rightarrow \mathbb{H}^2 / \Pi = \Sigma_g$ be the universal coverings. For two measured foliations $\lambda_+, \lambda_-$ on $\Sigma_g$, we set $\hat{\lambda}_+ = p_+^{-1}(\lambda_+) \subset \text{int} H_+$, $\hat{\lambda}_- = p_-^{-1}(\lambda_-) \subset \text{int} H_-$ and $\hat{\lambda}_+ = q^{-1}(\lambda_+)$, $\hat{\lambda}_- = q^{-1}(\lambda_-) \subset \mathbb{H}^2$. Consider the projection $\pi : \mathbb{B}^3 \rightarrow \mathbb{B}^2$ defined so that, for any $x \in \mathbb{B}^2 \subset \mathbb{B}^3$, $\pi(x) = x$, and for any $y \in \mathbb{B}^3 - \mathbb{B}^2$, $\pi(y)$ is the intersection point of $\text{int} \mathbb{B}^2 = \mathbb{H}^2$ with the geodesic line $l$ in $\mathbb{H}^3$ meeting $\mathbb{H}^2$ orthogonally and satisfying $\text{cl}(l) \ni y$, where $\text{cl}(l)$ is the closure of $l$ in $\mathbb{B}^3$. Then, we have $\pi(\hat{\lambda}_+) = \hat{\lambda}_+$ and $\pi(\hat{\lambda}_-) = \hat{\lambda}_-$.

Consider a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ with $\text{inj}(M) > 0$ and admitting an isomorphism $\rho : \Pi \rightarrow \Gamma \subset \text{Isom}^+(\mathbb{H}^3)$. According to Minsky [6, §7], if $M$ contains no geometrically finite ends, then there exist two measured foliations $\lambda_+, \lambda_-$ on $\Sigma_g$ and a $\rho$-equivariant, continuous map $F : \mathbb{B}^3 \rightarrow \mathbb{B}^3$ such that, for any leaves $l_+$ of $\hat{\lambda}_+$ and $l_-$ of $\hat{\lambda}_-$, $\text{cl}(l_+) \cap \text{cl}(l_-) = \emptyset$ and such that,
for any points \( x, y \in S_2^\infty \) with \( x \neq y \), \( F(x) = F(y) \) if and only if there exists a leaf \( l \) of either \( \lambda_+ \) or \( \lambda_- \) with \( \{x, y\} \subset \text{cl}(l) \).

The supports \( |\mu_+|, |\mu_-| \) of the measured laminations \( \mu_+, \mu_- \) on \( \Sigma_g \) corresponding to \( \lambda_+ \) and \( \lambda_- \) are called the ending laminations for \( M \). Since there exists the natural one-to-one correspondence between the set \( \mathcal{G} \) of leaves in \( \lambda_+ \cup \lambda_- \) not homeomorphic to the open interval and the set of connected components of \( (\text{int}^+ H_+ - p_+^{-1}(|\mu_+|)) \cup (\text{int}^- H_- - p_-^{-1}(|\mu_-|)), \mathcal{G} \) is a countable set. By [10, Proposition 9.3.8], each component of \( \Sigma_g - |\mu_+| \) and \( \Sigma_g - |\mu_-| \) is a finite-sided polygon with ideal vertices. It follows that, for each leaf \( l \) in \( \mathcal{G} \), \( \text{cl}(l) \cap S_1^\infty \) consists of finitely many points. We say that

\[
A_\Gamma = \{x \in S_1^\infty; x \in \text{cl}(l) \text{ for some } l \in \mathcal{G}\}
\]

is the countable, exceptional set for \( \Gamma \). Since \( F(S_1^\infty) \) is a \( \Gamma \)-invariant, closed subset of \( S_2^\infty \) and since the limit set of \( \Gamma \) is \( S_2^\infty \), \( F(S_1^\infty) \) coincides with \( S_2^\infty \). Thus, the restriction

\[
f = F|_{S_1^\infty}: S_1^\infty \rightarrow S_2^\infty
\]

is a \( \rho \)-equivariant, surjective, continuous map. We set

\[
\Lambda_1^\pm = \{x \in S_1^\infty; x \in \text{cl}(l) \text{ for some leaf } l \text{ of } \lambda_{\pm}\},
\]

and \( \Lambda^1 = \Lambda_1^+ \cup \Lambda_1^- \), \( \Lambda_2^1 = f(\Lambda_1^+) \), \( \Lambda_2^2 = f(\Lambda_1^-) = \Lambda_2^+ \cup \Lambda_2^- \).

The existence of such a map \( F \) was first shown by Cannon and Thurston [2] in special cases and by Minsky [6] for any \( M \) satisfying the above conditions.

2. Proof of Theorem

Let \( \mu_1, \mu_2 \) be respectively the 1-dimensional and 2-dimensional Lebesgue measures on \( S_1^\infty, S_2^\infty \) with respect to the fixed euclidean metrics on \( \mathbb{B}^2 \) and \( \mathbb{B}^3 \). The following lemma is the essential part of Theorem.

Lemma 1. \( \mu_2(\Lambda^2) = \mu_2(\Lambda_2^1) + \mu_2(\Lambda_2^-) = 0 \).

Proof. We will show that \( \mu_2(\Lambda_2^1) = 0 \). It is proved similarly that \( \mu_2(\Lambda_2^-) = 0 \).

Let \( \alpha_0 \) be a (short) geodesic segment in \( \mathbb{H}^2 \) meeting leaves of \( \lambda_+ \) transversely. If necessary replacing \( \alpha_0 \) by a sufficiently shorter subsegment, we may assume:

(2.1) For each leaf \( l \) of \( \lambda_+ \) meeting \( \alpha_0 \) non-trivially, \( l \cap \alpha_0 \) consists of a single point.

Consider the set \( X_0 \) of all points \( x \in \mathbb{B}^2 \) such that \( x \in \text{cl}(l) \) for some leaf \( l \) of \( \lambda_+ \) with \( l \cap \alpha_0 \neq \emptyset \). Note that \( X_0 \) is a closed (and hence compact) subset of \( \mathbb{B}^2 \); see Figure 1 on the next page.

Set \( \Pi = \{\gamma_0, \gamma_1, \gamma_2, \ldots\} \) so that \( \gamma_0 = 1 \), and let \( X_n = \gamma_n X_0 \), \( \alpha_n = \gamma_0 \alpha_0 \) for all \( n \in \mathbb{N} \). By [10, Proposition 9.3.8], for each leaf \( l \) of \( \lambda_+ \), the image \( q(l) \) is dense in \( \Sigma_g \). This shows that there exists \( \gamma_n \in \Pi \) such that \( \gamma_n^{-1} l \cap \alpha_0 \neq \emptyset \) or equivalently \( l \subset X_n \). Thus, we have \( \Lambda_1^1 \cup \mathbb{H}^2 = \bigcup_{n=0}^{\infty} X_n \). For each \( n \in \{0\} \cup \mathbb{N} \), \( Y_n = \pi^{-1}(X_n) \cap H_+ \) and \( \beta_n = \pi^{-1}(\alpha_n) \cap H_+ \) are homeomorphic respectively to \( X_n \) and \( \alpha_n \).

Since \( F(Y_n) \) is a compact (and hence closed) subset of \( S_2^\infty \), \( \Lambda_2^2 = \bigcup_{n=0}^{\infty} F(Y_n) \) is a \( \Gamma \)-invariant Borel set. Since the limit set of \( \Gamma \) is the whole sphere \( S_2^\infty \), by Sullivan [8] (see also Canary [1, §9]), there are no positive non-constant superharmonic functions on \( M \). Then, by Sullivan [9], the solid angle
\[ \sum_{\gamma \in \Gamma} \exp(-2\text{dist}(x_0, \gamma x_0)) \] of \( \Gamma \), \( x_0 \in \mathbb{H}^3 \), is infinite, and the action of \( \Gamma \) on \( S_\infty^2 \) is ergodic. This implies that either \( \mu_2(\Lambda_+^2) = 0 \) or \( \mu_2(S_\infty^2 - \Lambda_+^2) = 0 \). Here, we suppose that \( \mu_2(S_\infty^2 - \Lambda_+^2) = 0 \) and induce a contradiction. If \( \mu_2(F(Y_n)) = 0 \) for some \( n \in \{0\} \cup \mathbb{N} \), then for each \( m \in \{0\} \cup \mathbb{N} \), \( \mu_2(F(Y_m)) = \mu_2(\rho(\gamma^{-1}) F(Y_n)) = 0 \) and hence \( \mu_2(\Lambda_+^2) = 0 \). Thus, \( \mu_2(S_\infty^2 - \Lambda_+^2) = 0 \) implies that, for each \( n \in \{0\} \cup \mathbb{N} \), \( \mu_2(F(Y_n)) > 0 \). Since, by (2.1), the restriction \( F_n = F|_{\beta_n}: \beta_n \to F(\beta_n) = F(Y_n) \) is injective and since \( F_n \) is a closed map, \( F_n \) is a homeomorphism. We will define the \( \Gamma \)-invariant map \( \eta: S_\infty^2 \times S_\infty^2 \to \mathbb{R} \) as follows. Set \( \eta(x, y) = 0 \) if \( (x, y) \in S_\infty^2 \times S_\infty^2 - \Lambda_+^2 \times \Lambda_+^2 \). For \( (x, y) \in \Lambda_+^2 \times \Lambda_+^2 \), we set \( \eta(x, y) = \text{dist}_{\mathbb{H}^2}(l_x, l_y) \), where \( l_x, l_y \) are the leaves of \( \Lambda_+ \) containing \( l_x \subset \pi(F^{-1}(x)) \), \( l_y \subset \pi(F^{-1}(y)) \). Obviously, \( \eta \) is \( \Gamma \)-invariant. For any \( m, n \in \{0\} \cup \mathbb{N} \) (possibly \( m = n \)), we will show that the restriction \( \eta|_{F(Y_m) \times F(Y_n)} \) is a measurable function. By the continuities for \( F_m^{-1}: F(Y_m) \to \beta_m \) and \( F_n^{-1}: F(Y_n) \to \beta_n \), it is proved that

\[ \eta(x, y) = \text{dist}_{\mathbb{H}^2}(\pi(l_{F_m^{-1}(x)}), \pi(l_{F_n^{-1}(y)})) \]

is continuous in \( R_{m,n} = F(Y_m) \times F(Y_n) - F(A_T) \times F(Y_m) \cup F(Y_n) \times F(A_T) \), where \( l_{F_m^{-1}(x)}, l_{F_n^{-1}(y)} \) are the leaves of \( \Lambda_+ \) containing \( F_m^{-1}(x) \) and \( F_n^{-1}(y) \) respectively. In fact, for any \( (x, y) \in R_{m,n} \), there exist mutually disjoint 2-disks \( D_1, D_2 \) (resp. \( D_3, D_4 \)) in \( \mathbb{H}^2 \) which are closed neighborhoods of the end points of \( l_x \) with \( \alpha_m \cap (D_1 \cup D_2) = \emptyset \) (resp. \( \alpha_n \cap (D_3 \cup D_4) = \emptyset \)) and such that, in the case of \( x \neq y \),

\[ \text{dist}_{\mathbb{H}^2}((l_x \cup D_1 \cup D_2) \cap \mathbb{H}^2, (D_3 \cup D_4) \cap \mathbb{H}^2) \geq 2\eta(x, y) \]

\[ \text{dist}_{\mathbb{H}^2}((D_1 \cup D_2) \cap \mathbb{H}^2, (l_y \cup D_3 \cup D_4) \cap \mathbb{H}^2) \geq 2\eta(x, y) \].

For any point \( x' \) in a sufficiently small neighborhood \( U_x \) of \( x \) in \( F(Y_m) \), either \( l_x \) is homeomorphic to the closed interval or each branched point of \( l_x \) is contained in \( D_1 \cup D_2 \). Then, for any \( \varepsilon > 0 \), one can take \( U_x \) so small that, for any \( x' \in U_x \), \( l_{x'} - D_1 \cup D_2 \) is contained in the \( \varepsilon/2 \)-neighborhood of \( l_x - D_1 \cup D_2 \) in \( \mathbb{H}^2 \) and vice versa. We have the similar situation also in a small neighborhood of \( y \) in \( F(Y_n) \). This shows that \( \eta \) is continuous in \( R_{m,n} \). See Figure 2 for a typical example of the discontinuity for \( \eta \) in \( F(A_T) \times F(Y_n) \cup F(Y_m) \times F(A_T) \). In Figure 2, though \( \{x_n\} \subset S_\infty^2 \) converges to \( x \in F(A_T) \) and \( \{\eta(x_n, y)\} \) converges to \( s \), in general, \( \eta(x, y) = 1 \) does not coincide with \( s \).
Since $A_\Gamma$ is a countable set, for the product measure $\mu_2^2 = \mu_2 \times \mu_2$ on $S^2_\infty \times S^2_\infty$, $\mu_2^2(F(A_\Gamma) \times F(Y_n) \cup F(Y_m) \times F(A_\Gamma)) = 0$. This proves that $\eta|_{F(Y_m) \times F(Y_n)}$ is a measurable function. Since $\bigcup_{m,n=0}^{\infty} F(Y_m) \times F(Y_n) = \Lambda_+^2 \times \Lambda_+^2$ has full $\mu_2^2$-measure in $S^2_\infty \times S^2_\infty$, $\eta$ is a measurable function on $S^2_\infty \times S^2_\infty$. Since, noted as above, the solid angle of $\Gamma$ is infinite, by Sullivan [9, Theorem II], $\Gamma$ acts on $S^2_\infty \times S^2_\infty$ ergodically. Thus, there exists a subset $N$ of $S^2_\infty \times S^2_\infty$ with $\mu_2^2(N) = 0$ and such that $\lambda_1|_{S^2_\infty \times S^2_\infty - N}$ is a constant $R$. Note that there exists $X_n$ such that $\text{dist}_{H^3}(X_0, X_n) \geq R + 1$. Since $\mu_2^2(F(Y_0) \times F(Y_n)) > 0$, there exists $(y_0, y_n) \in F(Y_0) \times F(Y_n) - N$ such that $\eta(y_0, y_n) \geq R + 1$, a contradiction. Thus, we have $\mu_2(\Lambda_+^2) = 0$. This completes the proof.  

Here, it is worthwhile presenting an outline of the alternate proof of Lemma 1 given by the referee, which uses a little more information about the geometric model discussed in [2] and [6]. In fact, the referee proved that $\Lambda^2$ is in the complement of the conical limit set of $\Gamma$, and that set has zero-measure by Sullivan [9, p. 483, Corollary]. One can see this by considering the model metric $\sigma$ on $\tilde{\Sigma}_g \times \mathbb{R}$ given in [2] and [6] such that the universal cover $(\mathbb{H}^2 \times \mathbb{R}, \tilde{\sigma})$ is $\rho$-equivalently quasi-isometric to the hyperbolic space $\mathbb{H}^3$. In $(\mathbb{H}^2 \times \mathbb{R}, \tilde{\sigma})$, there are hyperplanes $l \times \mathbb{R}$ where $l$ is a leaf of either $\tilde{\lambda}_+$ or $\tilde{\lambda}_-$ which are totally geodesic, and map to quasi-geodesic planes in $\mathbb{H}^3$. The set $\Lambda^2$ is obtained as the images of the end points of $l \times \{0\}$ for all such leaves $l$. However, the ray $\{x\} \times [0, \infty)$ for $x \in l \subset \tilde{\lambda}_+$ (or $\{x\} \times (-\infty, 0]$ for $x \in l \subset \tilde{\lambda}_-$) also has the image terminating at the same point. This image $g$ is quasi-geodesic in $\mathbb{H}^3$, and $p(g)$ is within bounded distance of a geodesic leaving every compact set in $M = \mathbb{H}^3/\Gamma$, where $p: \mathbb{H}^3 \rightarrow M$ is the universal covering. Thus, the image point is non-conical.

Though the following lemma is probably well known or a folklore, the author does not know suitable references. For completeness, we will present the proof similar to that of Lemma 1.

**Lemma 2.** $\mu_1(\Lambda^1_+) = \mu_1(\Lambda^1_-) + \mu_1(\Lambda^1_1) = 0$.

**Proof.** Since $\Lambda^1_+$ is a $\Pi$-invariant, measurable set and since, by [9], $\Pi$ acts on $S^1_\infty$ ergodically, either $\mu_1(\Lambda^1_+) = 0$ or $\mu_1(S^1_\infty - \Lambda^1_1) = 0$. Here, we suppose
that $\mu_1(S_1^\infty - \Lambda_1^1) = 0$ and define the $\Pi$-invariant map $\xi: S_1^\infty \times S_1^\infty \to \mathbb{R}$ as follows. Set $\xi(x, y) = 0$ if $(x, y) \in S_1^\infty \times S_1^\infty - \Lambda_1^1 \times \Lambda_1^1$. Otherwise, $\xi(x, y) = \text{dist}_{H^2}(l_x, l_y)$, where $l_x, l_y$ are the leaves of $\lambda_+$ with $\text{cl}(l_x) \ni x$, $\text{cl}(l_y) \ni y$. The argument as in Lemma 1 shows that $\xi$ is a measurable function.

By the Hopf-Tsuji Theorem (see [9]), $\Pi$ acts on $S_1^\infty \times S_1^\infty$ ergodically. It follows that $\xi$ is constant almost everywhere, a contradiction. Thus, we have $\mu_1(\Lambda_1^1) = 0$ and similarly $\mu_1(\Lambda_1^2) = 0$. □

**Proof of Theorem.** It remains to prove that $f|_{S_1^\infty - \Lambda_1^1}: S_1^\infty - \Lambda_1^1 \to S_2^\infty - \Lambda_2^2$ is a homeomorphism. Since the restriction map is continuous and bijective, it suffices to show that it is a closed map. By the definition of relative topology, for any closed set $C$ in $S_1^\infty - \Lambda_1^1$, there exists a closed (and hence compact) set $C'$ in $S_1^\infty$ with $C = (S_1^\infty - \Lambda_1^1) \cap C'$. Then, $f(C) = f((S_1^\infty - \Lambda_1^1) \cap C') \subset f(S_1^\infty - \Lambda_1^1) \cap f(C') = (S_2^\infty - \Lambda_2^2) \cap f(C')$. For any $y \in (S_2^\infty - \Lambda_2^2) \cap f(C')$, there exists $x \in C'$ with $f(x) = y$. Since $x \notin f^{-1}(\Lambda_2^2) = \Lambda_2^1$, $x$ is contained in $C' \cap (S_1^\infty - \Lambda_1^1) = C$. This shows that $y \in f(C)$, or equivalently $f(C) = (S_2^\infty - \Lambda_2^2) \cap f(C')$. Since $f(C')$ is a compact (an hence closed) subset of $S_2^\infty$, $f(C)$ is a closed subset of $S_2^\infty - \Lambda_2^2$. This completes the proof. □

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