ON THE AREA DISTORTION BY QUASICONFORMAL MAPPINGS

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Abstract. We give the sharp constants in the area distortion inequality for quasiconformal mappings in the plane.

Astala [1] proved the following theorem conjectured by Gehring and Reich in [3]:

Theorem A. Let \( f \) be a \( K \)-quasiconformal mapping of \( D = \{ z : |z| < 1 \} \) onto itself with \( f(0) = 0 \). Then for any measurable \( E \subset D \) we have

\[ |f(E)| \leq C(K)|E|^{1/K}, \]

where \(| \cdot |\) stands for the area.

The first author [2] obtained a shorter proof which did not make use of the elaborate Thermodynamic Formalism and Holomorphic Motion Theory of the original proof of Astala. In late 1992 the second author [4] circulated a minimal proof which gives sharp bounds for the constants under the normalization \( f \in \Sigma(K) \), i.e. \( f \) is a \( K \)-quasiconformal mapping of the plane which is conformal on \( C \setminus \overline{D} \) and \( f(z) = z + o(1) \) near \( \infty \). In the interests of having a short sharp proof we combined our efforts.

Usually in what follows \( \Delta \) is the closed unit disk \( \{ z : |z| \leq 1 \} \), but any compact set of transfinite diameter 1 will do (and this is important in our proof). We note that this normalization implies that for any \( E \subset \Delta \) the area of \( E \) and \( f(E) \) is bounded by \( \pi \).

Theorem 1. Let \( f \) be a \( K \)-quasiconformal mapping of the plane which is conformal on \( C \setminus \Delta \), where \( \Delta \) is a compact set of transfinite diameter 1, and \( f(z) = z + o(1) \) near \( \infty \).

(i) If \( f \) is conformal on \( E \subset \Delta \) (i.e., \( f \) has dilatation \( \mu = 0 \) a.e. on \( E \)), then

\[ |f(E)| \leq \pi^{1-1/K}|E|^{1/K}. \]

(ii) If \( E \subset \Delta \) with \( f \) conformal on \( C \setminus E \), then

\[ |f(E)| \leq K|E|. \]
(iii) Hence in general for \( E \subset \Delta \)

\[
|f(E)| \leq K \pi^{1-1/K} |E|^{1/K}.
\]

**Remarks.** Theorem A follows from Theorem 1 via standard distortion estimates for quasiconformal mappings. The constants in Theorem 1 are best possible. Part (ii) is essentially due to Gehring and Reich. Part (i) gives sharp bounds for a conjectured inequality for the singular integral transform

\[
T g(\zeta) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|z-\zeta|>\varepsilon} \frac{g(z) dx \, dy}{(z-\zeta)^2},
\]

i.e., for every \( E \subset \Delta \) we have

\[
\int \int_{\Delta \setminus E} |T(\chi_E)| \, dx \, dy \leq |E| \log \frac{\pi}{|E|}.
\]

**Lemma 1.** Let \( a_1, \ldots, a_n \) be positive functions in the unit disk, such that \( \log a_j \) are harmonic and

\[
\sum_{j=1}^{n} a_j(\lambda) \leq 1, \quad |\lambda| < 1.
\]

Then

\[
\sum_{j=1}^{n} a_j(\lambda) \leq \left( \sum_{j=1}^{n} a_j(0) \right)^{(1-|\lambda|)/(1+|\lambda|)}, \quad |\lambda| < 1.
\]

The proof is based on the following "Variational Principle" from statistical mechanics which was also used by Astala.

**Lemma A.** Let \( p_1 > 0 \) and \( q_1 > 0 \) be probability distributions on the set \( \{1, \ldots, n\} \). Then

\[
-\sum_{j=1}^{n} p_j \log q_j + \sum_{j=1}^{n} p_j \log p_j \geq 0.
\]

**Proof.** The left side of the inequality is equal to \( \sum q_j \phi(p_j/q_j) \), where \( \phi(x) = x \log x \). This function \( \phi \) is convex, so

\[
\sum q_j \phi \left( \frac{p_j}{q_j} \right) \geq \phi \left( \sum q_j \frac{p_j}{q_j} \right) = \phi(1) = 0.
\]

**Proof of Lemma 1.** For \( |\lambda| < 1 \) and \( |z| < 1 \) define the probability distributions

\[
p_j = \frac{a_j(\lambda)}{\sum a_j(\lambda)} \quad \text{and} \quad q_j = \frac{a_j(z)}{\sum a_j(z)}.
\]

Now fix \( \lambda \) and set

\[
H(z) = -\sum p_j \log a_j(z) + \sum p_j \log p_j.
\]

Observe that \( H \) is harmonic in \( z \). By Lemma A and hypothesis (1)

\[
H(z) \geq -\log \sum a_j(z) \geq 0.
\]

Thus by Harnack's inequality

\[
H(z) \geq \frac{1 - |z|}{1 + |z|} H(0).
\]
Putting $z = \lambda$ and using Lemma A again we obtain
\[
H(\lambda) = -\log \sum a_j(\lambda) \geq \frac{1 - |\lambda|}{1 + |\lambda|} \left( -\log \sum p_j \log a_j(0) + \sum p_j \log p_j \right)
\geq \frac{1 - |\lambda|}{1 + |\lambda|} \left( -\log \sum a_j(0) \right),
\]
which proves Lemma 1.

Actually we require the continuous version of Lemma 1. Namely $a(z, \lambda)$ is to be defined on $E \times D$ and $\log a(z, \lambda)$ is harmonic in $\lambda$. If
\[
\int_E \int_E a(z, \lambda) dxdy \leq 1, \quad z = x + iy, \quad |\lambda| < 1,
\]
then we have
\[
\int_E \int_E a(z, \lambda) dxdy \leq \left( \int_E \int_E a(z, 0) dxdy \right)^{(1 - |\lambda|)/(1 + |\lambda|)}.
\]

The application to Theorem 1 is immediate. Suppose that $f$ has complex dilatation $\mu$ supported on $\Delta$. Without loss of generality we may assume that $\mu$ is smooth (a uniform bound for the smooth case yields the general uniform bound since the smooth case is dense). Define the function $f_\lambda \in \sum(K_\lambda)$, $K_\lambda = (1 + |\lambda|)/(1 - |\lambda|)$, with dilatation
\[
\mu_\lambda(z) = \frac{K + 1}{K - 1} \mu(z), \quad |\lambda| < 1.
\]
This is done by the standard solution of the Beltrami equation:
\[
f_\lambda(z) = z + S \mu_\lambda + S \mu_\lambda T \mu_\lambda + S \mu_\lambda T \mu_\lambda T \mu_\lambda + \cdots,
\]
where $S$ is the complex Cauchy transform. Now $f_\lambda$ has Jacobian
\[
J_\lambda(z) = |\partial_z f_\lambda(z)|^2 (1 - |\mu_\lambda(z)|^2).
\]
As the dilatations are smooth this is everywhere nonzero. If $f$ is conformal on $E$ define
\[
a(z, \lambda) = \frac{1}{\pi} |\partial_z f_\lambda(z)|^2.
\]
By the Holomorphic Dependence of Parameter Theorem for the Beltrami equation (see, for example, [5]) $\partial_z f_\lambda$ is holomorphic in $\lambda$. Thus $\log a(z, \lambda)$ is harmonic for $|\lambda| < 1$, $z \in E$. By the classical Area Theorem for a conformal mapping as $f_\lambda(z) = z + o(1)$, $z \to \infty$,
\[
\int \int_\Delta J_\lambda(z) dxdy \leq \pi.
\]
Thus $a(z, \lambda)$ satisfies the continuous version of Lemma 1 giving
\[
\int \int_E J_\lambda(z) \frac{dxdy}{\pi} \leq \left( \frac{|E|}{\pi} \right)^{(1 - |\lambda|)/(1 + |\lambda|)}.
\]
Setting $\lambda = (K - 1)/(K + 1)$ gives $\mu_\lambda = \mu$ and thus
\[
|f(E)| \leq \pi^{1 - 1/K} |E|^{1/K},
\]
completing the first part of the proof.
To prove part (ii) and the bound for $T$ we sketch the arguments of Gehring and Reich. This begins with the observation that for any set $G$

$$\int \int_G |T(\chi _G)| \, dx \, dy \leq |G|$$

(by Cauchy-Schwarz as $T$ is a unitary transformation of $L^2(\mathbb{C})$). Hence for any function $\rho$ supported on $G$ as $T$ is also (almost) self-adjoint

$$(2) \quad \left| \int \int_G T(\rho) \, dx \, dy \right| \leq \|\rho\|_\infty |G|.$$\n
Finally for any function $\mu$, $\|\mu\|_\infty = 1$, supported on $E$ we define $\mu_t(z) = t\mu(z)$ and the corresponding family of normalized maps $f_t$, $0 < t < 1$, $f_0(z) = z$ and $\int |\lambda| = f$. This can be realised as a deformation family of quasiconformal maps

$$\frac{\partial f_t}{\partial t} = g_t \circ f_t, \quad g_t(z) = z + S_{\rho_t},$$

$$\rho_t = \frac{\mu \circ f_t^{-1}}{1 - t^2 |\mu \circ f_t^{-1}|^2} e^{2i \arg (\partial_z f_t^{-1})}, \quad f_0(z) = z,$$

by the composition formula for dilatations. Now as $\partial S = T$

$$\frac{d|f_t(E)|}{dt} = 2\Re \int \int_{f_t(E)} T(\rho_t) \, dx \, dy.$$

Thus by (2)

$$\frac{d|f_t(E)|}{dt} \leq 2 \frac{|f_t(E)|}{1 - t^2},$$

so by integration

$$|f_t(E)| \leq \frac{1 + t}{1 - t^2} |E|,$$

which proves the result.

The third part follows by writing $f = g \circ h$ where $h$ is conformal on $E$ and $g$ is conformal on $\mathbb{C} \setminus h(E)$. Thus $h$ has dilatation $\mu(z)$ on $\Delta \setminus E$, zero elsewhere, and $g$ has dilatation $\mu(h^{-1}(z))$ on $h(E)$, zero elsewhere. We see that $h$ is normalized and so is $g$ as $h(\Delta)$ has transfinite diameter 1.

The bound on $T$ is also proved by holomorphic deformation. For any function $\mu$, $\|\mu\|_\infty < 1$, supported on $\Delta \setminus E$ we define $\mu_\lambda(z) = \lambda \mu(z)$ and the corresponding family of normalized maps $f_\lambda$. This time we let $\lambda \to 0$ to find that

$$|f_\lambda(E)| = |E| + 2\Re \left( \lambda \int \int_E T(\mu) \, dx \, dy \right) + o(\lambda)$$

$$\leq \pi^{2\lambda + o(\lambda)} |E|^{1 - 2\lambda + o(\lambda)} = |E| + 2|\lambda| |E| \log \frac{\pi}{|E|} + o(\lambda)$$

by part (i) of Theorem 1. Hence we obtain

$$\left| \int \int_E T(\mu) \, dx \, dy \right| \leq |E| \log \frac{\pi}{|E|}$$
and so as in the proof of (ii) for all \( \mu \) supported on \( \Delta \setminus E \) and bounded by 1

\[
\left| \int \int_{\Delta \setminus E} T(\chi_E) \bar{\mu}(z) \, dx \, dy \right| \leq |E| \log \frac{\pi}{|E|}.
\]

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