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NILPOTENCY OF DERIVATIONS IN PRIME RINGS

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ABSTRACT. In 1957, E. C. Posner proved that if λ and δ are derivations of a prime ring R, characteristic $R \neq 2$, then $\lambda\delta = 0$ implies either $\lambda = 0$ or $\delta = 0$. We extend this well-known result by showing that, without any characteristic restriction, $\lambda\delta^m = 0$ implies either $\lambda = 0$ or $\delta^{4m-1} = 0$. We also prove that $\lambda^n\delta = 0$ implies either $\delta^2 = 0$ or $\lambda^{12n-9} = 0$. In the case where $\lambda^n\delta^m = 0$, we show that if λ and δ commute, then at least one of the derivations must be nilpotent.

A derivation of a ring R is an additive map $\lambda: R \to R$ satisfying $\lambda(xy) = \lambda x \cdot y + x \cdot \lambda y$ for all $x, y \in R$. In 1957, Edward Posner showed that if λ and δ are derivations of a prime ring R, characteristic $R \neq 2$, then $\lambda\delta = 0$ implies either $\lambda = 0$ or $\delta = 0$ (see [3]). In this paper we show that in a prime ring with no characteristic restriction, $\lambda\delta = 0$ implies either $\lambda = 0$ or $\delta^2 = 0$. We generalize this result by proving that for any positive integer m, $\lambda\delta^m = 0$ implies either $\lambda = 0$ or $\delta^{4m-1} = 0$. Furthermore, we show that for any positive integer n, $\lambda^n\delta = 0$ implies either $\delta^2 = 0$ or $\lambda^{12n-9} = 0$. Lastly, in the general case where $\lambda^n\delta^m = 0$, we prove that if λ and δ commute, then at least one of the derivations must be nilpotent.

Lemma 1. *Assume λ is a derivation of a prime ring R and $\exists a \in R$, $a \neq 0$, such that $a(\lambda^n R) = 0$ or $(\lambda^n R)a = 0$ Then $\lambda^{2n-1} = 0$.*

Proof. Start by assuming $a(\lambda^n R) = 0$. Then for all $x, y \in R$, we have

$$a\lambda^n(xy) = a\left(\sum_{i=0}^{n}\binom{n}{i}\lambda^i x\lambda^{n-i}y\right) = 0. \tag{1}$$

Note that replacing x by $\lambda^{n-1}x$ in (1) yields $a\lambda^{n-1}x\lambda^n y = 0$. Replacing x by $\lambda^{n-2}x$ and y by λy in (1), and using the fact that $a\lambda^{n-1}x\lambda^n y = 0$, we get $a\lambda^{n-2}x\lambda^{n+1}y = 0$. At the next iteration we replace x by $\lambda^{n-3}x$ and y by $\lambda^2 y$ in (1), and use $a\lambda^{n-1}x\lambda^n y = 0$ and $a\lambda^{n-2}x\lambda^{n+1}y = 0$, to get $a\lambda^{n-3}x\lambda^{n+2}y = 0$. Continuing this process we eventually obtain $ax\lambda^{2n-1}y = 0$. Since this is true for all x and y in R, and R is a prime ring, we conclude that $\lambda^{2n-1} = 0$. Similarly, if we begin with $(\lambda^n R)a = 0$, we reach the same conclusion. □

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Lemma 2. *If λ and δ are derivations of a prime ring R and $\lambda\delta = 0$, then either $\lambda = 0$ or $\delta^2 = 0$.*

Proof. For all $x, y \in R$, we have $\lambda\delta(xy) = \lambda x\delta y + \delta x\lambda y = 0$. Replacing x by δx yields $\delta^2 x\lambda y = 0$. Apply Lemma 1. □

The distinction between Posner's result and Lemma 2 is made clear by a simple example. A derivation $\lambda: R \to R$ is called an inner derivation if there exists $a \in R$ such that $\lambda(x) = [a, x] = ax - xa$, for all $x \in R$. Let S be the 2×2 matrix ring over the Galois field $\{0, 1, w, w^2\}$, with inner derivations λ and δ defined by

$$\lambda(X) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X\right] \quad \text{and} \quad \delta(X) = \left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, X\right], \quad \forall X \in S.$$

The characteristic of S is 2, and we have $\lambda \neq 0$, $\delta \neq 0$, $\lambda\delta = 0$, and $\delta^2 = 0$.

Next we consider what can be said when λ and δ are derivations of a prime ring R and $\lambda^n\delta^m = 0$. The following three theorems assure us that at least one of the derivations must be nilpotent when $m = 1$, when $n = 1$, or when λ and δ commute.

Theorem 1. *Let λ and δ be derivations of a prime ring R, and let $\lambda\delta^m = 0$ where m is a positive integer. Then either $\lambda = 0$ or $\delta^r = 0$ where $r \leq 4m - 1$.*

Proof. We proceed by induction, noting that Lemma 2 implies the result is true for $m = 1$. Assume the theorem is true for $m = 1, 2, 3, \ldots, k - 1$. If $\lambda\delta^k = 0$, then for all x and y in R, $\lambda\delta^k(xy) = \lambda(\sum_{i=0}^{k} \binom{k}{i} \delta^{k-i} x \delta^i y) = 0$. Replacing x by $\delta^{k-1}x$ and y by $\delta^k y$ yields $\lambda(\delta^{k-1}x\delta^{2k}y) = \lambda\delta^{k-1}x\delta^{2k}y = 0$. Applying Lemma 1 gives the desired result. □

Theorem 2. *Let λ and δ be derivations of a prime ring R, and let $\lambda^n\delta = 0$ where n is a positive integer. Then either $\delta^2 = 0$ or $\lambda^t = 0$ where $t \leq 12n - 9$.*

Proof. The derivations of R form a Lie ring under commutation (see [2, p. 4]). Therefore $[\delta, \lambda] = \delta\lambda - \lambda\delta$ is a derivation, $[\delta\lambda - \lambda\delta, \lambda] = \delta\lambda^2 - 2\lambda\delta\lambda + \lambda^2\delta$ is a derivation, and $[\delta\lambda^2 - 2\lambda\delta\lambda + \lambda^2\delta, \lambda] = \delta\lambda^3 - 3\lambda\delta\lambda^2 + 3\lambda^2\delta\lambda - \lambda^3\delta$ is a derivation. Continuing we may conclude that $\sum_{i=0}^{2n-1} \binom{2n-1}{i} (-1)^i \lambda^i \delta \lambda^{2n-1-i}$ is a derivation. The coefficients are not germane to the rest of the proof, so we suppress them from here on out. Using the fact that $\lambda^n\delta = 0$ we get $\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \cdots + \lambda^{n-1}\delta\lambda^n$ is a derivation. Applying Lemma 2 to $(\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \cdots + \lambda^{n-1}\delta\lambda^n)\delta = 0$ yields $\delta^2 = 0$ or

$$\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \cdots + \lambda^{n-1}\delta\lambda^n = 0. \tag{2}$$

If $\delta^2 \neq 0$, then premultiply (2) by λ^{n-1} to obtain $\lambda^{n-1}\delta\lambda^{2n-1} = 0$. Premultiplying (2) by λ^{n-2} it follows that

$$\begin{aligned} &\lambda^{n-2}\delta\lambda^{2n-1} + \lambda^{n-1}\delta\lambda^{2n-2} = 0 \\ &\quad\Rightarrow (\lambda^{n-2}\delta\lambda^{2n-1} + \lambda^{n-1}\delta\lambda^{2n-2})\lambda = 0 \\ &\quad\Rightarrow \lambda^{n-2}\delta\lambda^{2n} = 0. \end{aligned}$$

Premultiplying (2) by λ^{n-3} it follows that

$$\begin{aligned}&\lambda^{n-3}\delta\lambda^{2n-1}+\lambda^{n-2}\delta\lambda^{2n-2}+\lambda^{n-1}\delta\lambda^{2n-3}=0\\&\quad\Rightarrow(\lambda^{n-3}\delta\lambda^{2n-1}+\lambda^{n-2}\delta\lambda^{2n-2}+\lambda^{n-1}\delta\lambda^{2n-3})\lambda^{2}=0\\&\quad\Rightarrow\lambda^{n-3}\delta\lambda^{2n+1}=0.\end{aligned}$$

Continuing this algorithm we eventually arrive at $\delta\lambda^{3n-2}=0$. Applying Theorem 1 finishes the proof. □

Lung Chung and Jiang Luh have shown that in a prime ring with characteristic 2, the nilpotency of a nilpotent derivation must be of the form 2^k (see [1]). Therefore, when R is not 2-torsion free, the possible values for nilpotency in Theorems 1 and 2 are further limited. For example, if we assume in Theorem 2 that characteristic $R=2$, $\lambda^6\delta=0$, and $\delta^2\neq 0$, then the nilpotency of λ must be $1, 2, 4, 8, 16$, or 32.

Theorem 3. *Assume λ and δ are derivations of a prime ring R, and assume $\lambda^n\delta^m=0$ where n and m are positive integers. If λ and δ commute, then at least one of them is nilpotent.*

Proof. From the hypotheses we know that for all $x, y\in R$,

$$\begin{aligned}&\lambda^n\delta^m(\delta^{m-1}\lambda^n x\lambda^{n-1}y)=\lambda^n(\delta^{m-1}\lambda^n x\delta^m\lambda^{n-1}y)=\delta^{m-1}\lambda^{2n}x\delta^m\lambda^{n-1}y=0\\&\quad\Rightarrow\lambda^n\delta^m(\delta^{m-2}\lambda^{2n}x\delta^m\lambda^{n-1}y)=\lambda^n(\delta^{m-2}\lambda^{2n}x\delta^{2m}\lambda^{n-1}y)\\&\qquad\qquad=\delta^{m-2}\lambda^{3n}x\delta^{2m}\lambda^{n-1}y=0\\&\quad\Rightarrow\lambda^n\delta^m(\delta^{m-3}\lambda^{3n}x\delta^{2m}\lambda^{n-1}y)=\lambda^n(\delta^{m-3}\lambda^{3n}x\delta^{3m}\lambda^{n-1}y)\\&\qquad\qquad=\delta^{m-3}\lambda^{4n}x\delta^{3m}\lambda^{n-1}y=0\\&\quad\vdots\\&\quad\Rightarrow\lambda^n\delta^m(\delta\lambda^{(m-1)n}x\delta^{(m-2)m}\lambda^{n-1}y)=\lambda^n(\delta\lambda^{(m-1)n}x\delta^{(m-1)m}\lambda^{n-1}y)\\&\qquad\qquad=\delta\lambda^{mn}x\delta^{(m-1)m}\lambda^{n-1}y=0\\&\quad\Rightarrow\lambda^n\delta^m(\lambda^{mn}x\delta^{(m-1)m}\lambda^{n-1}y)=\lambda^n(\lambda^{mn}x\delta^{m^2}\lambda^{n-1}y)\\&\qquad\qquad=\lambda^{(m+1)n}x\delta^{m^2}\lambda^{n-1}y=0.\end{aligned}$$

Using Lemma 1, we have $\lambda^{2(m+1)n-1}=0$ or $\lambda^{n-1}\delta^{m^2}=0$. If $\lambda^{2(m+1)n-1}\neq 0$, we then apply the above process to $\lambda^{n-1}\delta^{m^2}=0$ to get $\lambda^{2(m^2+1)(n-1)-1}=0$ or $\lambda^{n-2}\delta^{m^4}=0$. If λ is not nilpotent, we continue and eventually arrive at $\delta^{m^{2n}}=0$. □

As a final note, if λ, δ, and γ are derivations of a prime ring R and $\lambda^n\delta^m\gamma^h=0$, it does not follow that one of the derivations must be nilpotent. For example, let T be the 3×3 matrix ring over a division ring D, and let λ, δ, and γ be the inner derivations defined by the unit matrices E_{11}, E_{22}, and E_{33}, respectively. Then the derivations commute with each other, and $\lambda\delta\gamma=0$. However, none of the derivations are nilpotent.

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