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ON RESIDUALLY FINITE-DIMENSIONAL C^*-ALGEBRAS

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ABSTRACT. Exel and Loring have listed several conditions that are equivalent to the residual finite-dimensionality of a C^*-algebra. We review and extend this list.

A C^*-algebra is said to be *residually finite-dimensional* (RFD) if it has a separating family of finite-dimensional representations. Goodearl and Menal [8] have shown that every C^*-algebra is the homomorphic image of an RFD C^*-algebra. Earlier, Choi [4] had shown that the full C^*-algebra of the free group on two generators is RFD. The connections between freeness and the RFD property have been developed in [8, 6, 10]. In the course of this, Exel and Loring gave several equivalent conditions for the RFD property [6, Theorem 2.4]. The purpose of this note is two-fold: to give some further equivalent conditions and to show how the main step in [6, Theorem 2.4] can be related to a result of Bichteler [3].

Let A be a C^*-algebra and H a Hilbert space. We denote by $\mathrm{Rep}(A, H)$ the set of all (possibly degenerate) representations of A on H, with the topology of pointwise strong-operator convergence. A representation π of A on a Hilbert space H_π is said to be *finite-dimensional* if its essential subspace (the closure of $\pi(A)H_\pi$) is finite-dimensional. Note that the finite-dimensionality of $\pi(A)$ is necessary, but in general not sufficient, for the finite-dimensionality of π. The representation π is said to be *residually finite-dimensional* if it lies in the closure of the set of finite-dimensional representations in $\mathrm{Rep}(A, H_\pi)$.

Let $S(A)$ be the state space of the C^*-algebra A, and let $P(A)$ and $F(A)$ be the sets of pure states and factorial states respectively. A state ϕ of A is said to be *finite-dimensional* if the Gelfand-Naimark-Segal representation π_ϕ is finite-dimensional (or, equivalently, $\pi_\phi(A)$ is finite-dimensional). The (possibly empty) set of finite-dimensional states of A is denoted by $\mathrm{Fin}(A)$ (in [6] the notation $F(A)$ is used for this, but we prefer to reserve $F(A)$ for the factorial states). As noted in [6], $\mathrm{Fin}(A)$ is a convex subset of $S(A)$.

Theorem (see [6, Theorem 2.4]). *Let A be a C^*-algebra. The following conditions are equivalent*:

(a) $\mathrm{Fin}(A)$ *is w^*-dense in the state space $S(A)$.*

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(b) *Every cyclic representation of A is residually finite-dimensional.*
(c) *Every representation of A is residually finite-dimensional.*
(d) *There exists a faithful residually finite-dimensional representation of A.*
(e) *A is residually finite-dimensional.*
(f) *The set of finite-dimensional irreducible representations of A is dense in the spectrum $\widehat{A}$ of A.*
(g) *$P(A) \cap \mathrm{Fin}(A)$ is w^*-dense in $P(A)$.*
(h) *$F(A) \cap \mathrm{Fin}(A)$ is w^*-dense in $F(A)$.*

Proof. The equivalence of the first five conditions is proved in [6, Theorem 2.4]. However, the main step ((a) $\Rightarrow$ (b)) may also be obtained from a related result of Bichteler as follows.

(a) $\Rightarrow$ (b) Let H be a Hilbert space with infinite dimension that is large enough to ensure that every cyclic representation of A is unitarily equivalent to some representation on a closed subspace of H. Suppose that π is an infinite-dimensional cyclic representation of A, and let H_π be regarded as a subspace of H. Let $\varepsilon > 0$, $a_1, \dots, a_n \in A$ and $\xi_1, \dots, \xi_m \in H_\pi$. We seek a finite-dimensional representation ρ of A on H_π such that

$$\|\rho(a_i)\xi_j - \pi(a_i)\xi_j\| < \varepsilon \qquad (1 \le i \le n, \ 1 \le j \le m).$$

Let $\xi \in H_\pi$ be a unit vector that is cyclic for π, and define $\phi = \langle \pi(\cdot)\xi, \xi\rangle \in S(A)$. By [3, p. 92, Proposition] and (a), there exist $\psi \in \mathrm{Fin}(A)$ and a representation σ of A on H such that π_ψ is unitarily equivalent to the restriction of σ to its essential subspace H_σ and

$$\|\sigma(a_i)\xi_j - \pi(a_i)\xi_j\| < \varepsilon \qquad (1 \le i \le n, 1 \le j \le m).$$

Let K be the closed subspace of H generated by H_π and H_σ. Since H_π is infinite-dimensional and H_σ is finite-dimensional, H_π and K have the same Hilbert dimension. Hence there exists a unitary operator U from H_π onto K such that U fixes every element in the (finite-dimensional) linear span of the set $\{\xi_j, \pi(a_i)\xi_j \colon 1 \le i \le n, \ 1 \le j \le m\}$. We define a finite-dimensional representation ρ of A on H_π by setting $\rho(a) = U^*\sigma(a)U$ $(a \in A)$. Then for $1 \le i \le n$ and $1 \le j \le m$ we have

$$\|\rho(a_i)\xi_j - \pi(a_i)\xi_j\| = \|U^*\sigma(a_i)\xi_j - U^*\pi(a_i)\xi_j\| < \varepsilon$$

as required.

(e) $\Rightarrow$ (f) This follows from the fact that every non-degenerate finite-dimensional representation of A is a direct sum of finite-dimensional irreducible representations.

(f) $\Rightarrow$ (g) This follows from the open property of the canonical mapping from $P(A)$ onto the spectrum $\widehat{A}$ [5, 3.4.11].

(g) $\Rightarrow$ (h) Let $\mathscr{U}$ be the unitary group of A (if A is unital) or of $A + \mathbb{C}1$ (if A is non-unital and 1 is an adjoined identity). Let

$$\psi = \sum_{i=1}^{k} \lambda_i \phi(u_i^* \cdot u_i)$$

where $\phi \in P(A)$, $k \in \mathbb{P}$, $\lambda_i \ge 0$ $(1 \le i \le k)$, $\sum_{i=1}^{k} \lambda_i = 1$ and $u_1, \dots, u_k \in \mathscr{U}$. Since the set of all such ψ is a w^*-dense subset of $F(A)$ [2, Proposition 2.2], it suffices to show that ψ lies in the w^*-closure of $F(A) \cap \mathrm{Fin}(A)$.

Assuming (g), there is a net (ϕ_a) in $P(A) \cap \mathrm{Fin}(A)$ such that $\phi_\alpha \to \phi$. For each α let

$$\psi_\alpha = \sum_{i=1}^{k} \lambda_i \phi_\alpha(u_i^* \cdot u_i).$$

Then $\psi_\alpha \in F(A) \cap \mathrm{Fin}(A)$ (since $\mathrm{Fin}(A)$ is convex and saturated with respect to unitary equivalence) and $\psi_\alpha \to \psi$.

(h) $\Rightarrow$ (a) Assuming (h), and using the convexity of $\overline{\mathrm{Fin}(A)}$, we have

$$S(A) \subseteq \overline{\mathrm{co}(P(A))} \subseteq \overline{\mathrm{co}(F(A))} \subseteq \overline{\mathrm{Fin}(A)}$$

(where co denotes convex hull and the closures are taken in the w^*-topology). □

Remarks. (i) The construction of ρ from σ in (a) $\Rightarrow$ (b) is similar to methods in [7].

(ii) The implication (g) $\Rightarrow$ (a) may of course be proved directly by an argument similar to that used for (h) $\Rightarrow$ (a).

(iii) Pestov [10] has recently used the equivalence of (c) and (e). Spaces $\mathrm{Rep}(A, H)$ have also been used recently in [1, 9].

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