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MARKOV'S EXPONENT OF COMPACT SETS IN $\mathbb{C}^n$

M. BARAN AND W. PLEŚNIAK

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ABSTRACT. We introduce the notion of Markov's exponent of a compact set in $\mathbb{C}^n$ and show that it is invariant under regular analytic maps.

1. MARKOV'S INEQUALITY

Given a compact subset E of the space $\mathbb{C}^n$ and a number $r \geq 1$, consider the following two conditions.

$M(r)$ There exists a constant $M_1 > 0$ such that for each $p \in \mathscr{P}_k$, $k = 1, 2, \dots$,

$$\|\operatorname{grad} p\|_E \leq M_1 k^r \|p\|_E.$$

$P(r)$ There exist two positive constants M_2 and C_2 such that for each $p \in \mathscr{P}_k$, $k = 1, 2, \dots$,

$$|p(x)| \leq M_2 \|p\|_E, \text{ as } \operatorname{dist}(x, E) \leq \frac{C_2}{k^r}.$$

Here $\mathscr{P}_k$ denotes the space of all polynomials of degree at most k. The condition $M(r)$ is a multidimensional version of the well-known inequality, proved by A.A. Markov in 1889 in the case where $E = [-1, 1]$. For the proof of this famous result and its one-dimensional generalizations we refer the reader to [RS]. Some criteria for subsets of $\mathbb{C}^n$ satisfying $M(r)$ have been proved in [PP1], [B1] and [B2]. In particular, it is known that every fat subanalytic set and, more generally, every uniformly polynomially cuspidal set satisfies $M(r)$, for some $r \geq 1$. Markov's inequality has been applied in problems connected with approximation and extension of $\mathscr{C}^\infty$ functions (see [PP2] and [Pl3]). Given a compact set E in $\mathbb{C}^n$, an important point is to determine the minimal constant r in $M(r)$. This permits, in particular, the minimization of the loss of regularity in problems connected with the linear extension of classes of $\mathscr{C}^\infty$ functions with restricted growth of derivatives (see [PS] and [Pl4]). In the next section we call such an r Markov's exponent of E and show that it is invariant under regular holomorphic maps. We close this section by proving the following observation.

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Proposition 1.1. *For each* $r \geq 1$, *the properties* $M(r)$ *and* $P(r)$ *of the set* E *are equivalent.*

Proof. Assume $M(r)$. Fix $a \in E$, $v \in \mathbb{C}^n$ with $\|v\| = 1$, and $p \in \mathscr{P}_k$, and define $q(t) = p(a + tv)$. Observe that $q^{(j)}(t) = D_v^j p(a + tv)$. In particular, we have $q^{(j)}(0) = D_v^j p(a)$. Hence, if $t \in \mathbb{C}$ and $|t| \leq C_2/k^r$, we can write

$$|q(t)| = |\sum_{j=0}^{k} \frac{1}{j!} t^j q^{(j)}(0)| \leq \sum_{j=0}^{k} \frac{1}{j!} |t|^j |D_v^j p(a)| \leq \sum_{j=0}^{k} \frac{1}{j!} M_1^j k^{jr} |t|^j \|p\|_E \leq M_2 \|p\|_E.$$

The inverse implication follows easily from Cauchy's integral formula. The proof is completed.

2. Markov's exponent

Given a compact subset E of the space $\mathbb{C}^n$, we define

$$\mu(E) = \inf\{r : E \text{ satisfies } M(r)\}$$

and call this number *Markov's exponent* of E. If E is a continuum in $\mathbb{C}$ containing at least two different points, then by [Po], $1 \leq \mu(E) \leq 2$. For any compact subset E of $\mathbb{R}^n$, we have $\mu(E) \geq 2$. If E is a fat convex subset of $\mathbb{R}^n$, then $\mu(E) = 2$. If $E = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x^p\}$, for $p \geq 1$, then by [G] $\mu(E) = 2p$. In a more general case, if E is an $m - UPC$ subset of $\mathbb{R}^n$ ($m \geq 1$) (see [PP1]), then by [B2] $\mu(E) \leq 2m$. If $E = \{(x, y) \in \mathbb{R}^n : 0 < x \leq 1, \quad 0 < y \leq e^{-1/x}\} \cup \{(0, 0)\}$, then by [Z] $\mu(E) = \infty$. The following example is a slight generalization of the above-mentioned results of Goetgheluck and Zerner.

Example 2.1. Let ϕ be a convex, increasing $\mathscr{C}^1$ function defined on $[0, 1]$ such that $\phi(0) = \phi'(0) = 0$, $\phi(1) = 1$. Define

$$E = E_\phi = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq \phi(1 - |x|)\}.$$

Then, for any polynomial $p \in \mathscr{P}_k$, $k = 0, 1, \ldots$, we have

$$\|\frac{\partial p}{\partial x}\|_E \leq \phi'(1) k^2 \|p\|_E. \tag{2.1.1}$$

Moreover, let

$$\alpha = \liminf_{t \to 0_+} \frac{\log \phi(t)}{\log t}, \quad \beta = \limsup_{t \to 0_+} \frac{\log \phi(t)}{\log t}.$$

Then, if $\beta < \infty$, for any $\epsilon > 0$ we have

$$\|\frac{\partial p}{\partial y}\|_E \leq \text{const.}\, k^{2(\beta+\epsilon)} \|p\|_E, \tag{2.1.2}$$

whence $\mu(E) \leq 2\beta$ ($\mu(E) = 2$, if $\beta = 1$). If $\alpha = \infty$, then

$$\mu(E) = \infty. \tag{2.1.3}$$

If ϕ satisfies Orlicz's Δ_2 condition at 0 (i.e., $\phi(2u) \leq \text{const.}\, \phi(u)$), then $\beta < \infty$ and ϕ does not satisfy this condition in case $\alpha = \infty$.

Proof. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and let

$$\rho_i(x, y) = \text{dist}_{e_i}((x, y), \mathbb{R}^2 \setminus E)$$

be the distance of $(x, y) \in E$ from the boundary of E in direction of the vector e_i, for $i = 1, 2$. Then one can easily check that

$$\rho_1(t(x, y)) \geq \frac{1}{\phi'(1)}(1 - |t|) \tag{2.1.4}$$

and

$$\rho_2(t(x, y)) \geq \phi(1 - |t|) \tag{2.1.5}$$

for $t \in [-1, 1]$ and $(x, y) \in \partial E$. Now, by (2.1.4), applying a version of Markov's inequality for star-shaped sets (see [B2, Thm. 3.6]) gives (2.1.1) (cf. [B2]). If $\beta < \infty$, then, for each $\epsilon > 0$, $\phi(t) \geq \text{const.}\, t^{\beta+\epsilon}$ for $t \in [0, 1]$. Hence by (2.1.5) and by [B2, Thm. 3.6] we get (2.1.2). Following an idea of Zerner [Z], suppose now that $\alpha = \infty$. Then, for each $r > 0$, we have $\frac{\log \phi(t)}{\log t} \geq r$, if $0 < t \leq \delta = \delta(r)$; whence $\phi(t) \leq Mt^r$ for $0 \leq t \leq 1$, where $M > 0$ is a constant depending on r. Now, if we take $p_k(x, y) = x^k y$, then $\|\frac{\partial p_k}{\partial y}\|_E = 1$ and

$$\begin{aligned}\|p_k\|_E = \sup_{|x|\leq 1} |x|^k \phi(1 - |x|) &\leq M \sup_{0\leq t\leq 1} t^k(1-t)^r = M\left(\frac{k}{k+r}\right)^k\left(\frac{r}{k+r}\right)^r \\ &\leq Mr^{r-1}\frac{1}{(k+1)^r} = M_1(r)\frac{1}{(k+1)^r}.\end{aligned}$$

Consequently,

$$\|\frac{\partial p_k}{\partial y}\|_E \geq M_2(r)(k+1)^r \|p_k\|_E$$

which shows that E cannot have Markov's property. Let

$$a_\phi^0 = \liminf_{t\to 0+} \frac{t\phi'(t)}{\phi(t)}, \quad b_\phi^0 = \limsup_{t\to 0+} \frac{t\phi'(t)}{\phi(t)}$$

be the lower and upper Simonenko indices at 0, respectively (see [M]). By Cauchy's mean value theorem

$$a_\phi^0 \leq \alpha \leq \beta \leq b_\phi^0,$$

and the relation between α, β and the Δ_2 condition follows from [M, Thm. 3.2(b)].

Consider e.g. $\phi(t) = t^p$, $p > 1$. Then $\phi'(1) = p$, $\alpha = \beta = p$. If $\phi(t) = e^{2(1-t^{-1})}$, then $\phi'(1) = 2$ and $\alpha = \beta = \infty$. If $\phi(t) = t(1 - \log t)^{-1}$, then $\phi'(1) = 2$ and $\alpha = \beta = 1$.

To prove the invariance of Markov's exponent under analytic mappings we need the following.

Lemma 2.2. *Let E be a polynomially convex, compact subset of $\mathbb{C}^n$ satisfying $M(r)$. Let f be a holomorphic mapping defined in a neighbourhood U of E, with values in $\mathbb{C}^m$, such that $f(E)$ is not pluripolar. Then there exist positive constants M_2 and C_3 such that for each polynomial $p \in \mathscr{P}_k(\mathbb{C}^m)$ and $k = 1, 2, \ldots$, we have*

$$|(p \circ f)(x)| \leq M_2 \|p \circ f\|_E \quad \text{as } \operatorname{dist}(x, E) \leq \frac{C_3}{k^r}.$$

Proof. Without loss of generality we may asuume that f is bounded on U. Since E is polynomially convex, one can find a compact polynomial polyhedron P such that $E \subset \operatorname{int} P \subset P \subset U$. By a uniform version of the Bernstein-Walsh-Siciak theorem (see [Pl1]), there exist constants $A > 0$ and $a \in (0, 1)$ such that for any polynomial $p \in \bigcup_{k=0}^{\infty} \mathscr{P}_k(\mathbb{C}^m)$, we have

$$\text{(2.2.1)} \qquad \operatorname{dist}_P(p \circ f, \mathscr{P}_l(\mathbb{C}^n)) \le A a^l \|p \circ f\|_U.$$

Since $f(E)$ is not pluripolar, we have $\|p \circ f\|_U = \|p\|_{f(U)} \le \|p\|_{f(E)} B^k$, where $B = \sup\{\Phi_{f(E)}(w) : w \in f(U)\} < \infty$ and where $\Phi_{f(E)}$ denotes Siciak's extremal function associated with $f(E)$ ([S1],[S2]). Choose k_0 such that $E_k(C_2, r) := \{x \in \mathbb{C}^n : \operatorname{dist}(x, E) \le C_2/k^r\} \subset P$ for $k \ge k_0$. Let $s \in \mathbb{N}$ be so large that $a^s B \le 1$, and let q_k be a best approximation polynomial to $p \circ f$ of degree $l = sk$. Then by (2.2.1), since E satisfies $P(r)$, we can write

$$|(p \circ f)(x)| \le A a^{sk} B^k \|p \circ f\|_E + |q(x)| \le A\|p \circ f\|_E + M_2\|q\|_E \le (A + 2M_2)\|p \circ f\|_E$$

as $\operatorname{dist}(x, E) \le C_2/(s^r k^r)$, which gives the lemma with $C_3 = C_2/s^r$ and $M_3 = A + 2M_2$.

Corollary 2.3. *Under the assumptions of Lemma 2.2, there exists a positive constant M_3' such that for each $v \in \mathbb{C}^m$ with $\|v\| = 1$, we have*

$$\|D_v(p \circ f)\|_E \le M_3' k^r \|p \circ f\|_E.$$

Proof. Fix $a \in E$ and $v \in \mathbb{C}^m$ with $\|v\| = 1$, and define $g(t) := (p \circ f)(a + tv)$. By Cauchy's integral formula, for $\delta = C_3/k^r$, we get

$$|D_v(p \circ f)(a)| = |g'(0)| \le \sup\{|g(\zeta)| : |\zeta| = \delta\}/\delta \le (M_2/C_3)k^r \|p \circ f\|_E.$$

Remark 2.4. The assumption that $f(E)$ is not pluripolar yields immediately the restrictions that $m \le n$ and f is non-degenerate at least in one of the connected components of U, say V, that meets the set E, which means that $\sup_{x \in V} \operatorname{rank}_x f = m$. If we knew that the Markov property of E implies that E is not pluripolar, we could replace the above assumption on $f(E)$ by the requirement that f is non-degenerate on at least one of the connected components of U that meet E at a non-pluripolar set. This, however, still seems to be unknown except when E is a Cantor type subset of $\mathbb{R}$ (see [Pl2],[BC]).

Lemma 2.2 together with Proposition 1.1 permits us to give a "sharp" version of Proposition 4.1 in [Pl3] by showing that Markov's exponent of a compact set is invariant under holomorphic injections. More precisely, we have the following

Theorem 2.5. *Under the assumptions of Lemma 2.2, suppose that $m = n$ and $\det d_x f \ne 0$ for each $x \in E$. Then $f(E) \in M(r)$.*

Proof. Choose $c > 0$ so that $|J_{\mathbb{C}} f(x)|^2 \ge c$. By the assumptions and the implicit function theorem there exist positive constants L and L_1 such that for each $x \in E$ and each $\delta \in (0, c]$, $f(B(x, L\delta)) \supset B(f(x), L_1\delta)$ (see [T, Chap. I, Prop. 5.1]). Choose $k_0 \in \mathbb{N}$ so that $C_3/Lk^r \le c$ for $k \ge k_0$. Then by Lemma 2.2, for a fixed $b \in f(E)$ and $a \in f^{-1}(\{b\})$, we get

$$|p(w)| \le M_2 \|p\|_{f(E)} \text{ as } |w - b| \le L_1 C_3/Lk^r,$$

for any polynomial $p \in \mathscr{P}_k(\mathbb{C}^m)$ and $k \geq k_0$. Since $f(E)$ is not pluripolar, the last inequality holds for any $k \in \mathbb{N}$ after a suitable change of the constant M_2. In view of Proposition 1.1, the proof of the theorem is complete.

Remark 2.6. Note that for any compact subset E of $\mathbb{C}^n$, E satisfies $M(r)$ iff $\hat{E} \in M(r)$ where $\hat{E}$ denotes the polynomial hull of E. However, the assumption that $E = \hat{E}$ cannot be removed. To see why, take E to be the set $\{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$ and $f(z) = 1/(z - 1/2)$. Choose a polynomial p such that $\|p\|_{f(\{|z|=1\})} \leq 1$ and $|p(-2)| \geq 2$. Then, if $f(E)$ satisfied $\mathscr{M}(r)$ for a certain $r > 0$, we would have, for $q_n(z) = (z+2)p^n(z)$, $2^n \leq |q_n'(-2)| \leq M(nd+1)^r\|q\|_{f(E)} \leq 4M(nd+1)^r$, for $n = 1, 2, \ldots$ with a positive constant M independent of n and where d denotes the degree of p, a contradiction.

Suppose now E is a compact subset of the space $\mathbb{R}^n$. (Here we assume that $\mathbb{R}^n$ is a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{R}^n + i\,\mathbb{R}^n$.) Suppose moreover E is a UPC set (cf. [PP1]). This means that there exist positive constants M and m, a positive integer d and a mapping $h : E \times [0, 1] \to E$ such that for each $x \in E$, $h(x, \cdot)$ is a polynomial of degree at most d, $h(x, 1) = x$ and $\mathrm{dist}(h(x, t), \mathbb{R}^n \setminus E) \geq M(1-t)^m$ for all (x, t) in $E \times [0, 1]$. It was shown in [PP2] that if f is a $\mathscr{C}^\infty$ mapping defined in $\mathbb{R}^n$ with $J_{\mathbb{R}}f(x) \neq 0$ on E, then $f(E)$ also is a UPC subset of $\mathbb{R}^n$, whence a Markov set. If we drop the assumption that $\det d_x f \neq 0$ everywhere on E, the above theorem fails to hold, which is seen by the following

Counter-example 2.7. Take E to be the set $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\}$. Then E satisfies $M(2)$, since it is convex. Consider now the map $f(x, y) = (x, y\phi(x))$ for $(x, y) \in \mathbb{R}^2$, where

$$\phi(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a $\mathscr{C}^\infty$ mapping on $\mathbb{R}^2$ with $\det d_{(x,y)}f \neq 0$ on $E \setminus \{(0, 0)\}$. Nevertheless, the set $f(E) = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq 1,\ 0 \leq v \leq u\phi(u)\}$ is known to not preserve Markov's inequality for any order $r > 0$.

The situation is much better when f is a polynomial. We have

Theorem 2.8. *Let E be a compact subset of $\mathbb{R}^n$ that is UPC with parameters $M > 0$, $m \geq 1$ and $d \in \mathbb{N}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map of degree l with $\det d_x f \neq 0$ on $\mathrm{int}\, E$. Then there exist constants $M_0 > 0$ and $m_0 \geq 1$ such that*

$$\|(d_x f)^{-1}\| \leq M_0(\mathrm{dist}(x, \partial E))^{-m_0} \tag{2.8.1}$$

and for any polynomial $p \in \mathscr{P}_k$, $k = 0, 1, \ldots,$

$$\|\mathrm{grad}\ p\|_{f(E)} \leq Ck^{2m(m_0+1)}\|p\|_{f(E)} \tag{2.8.2}$$

with $C = 2M_0M^{-(mm_0+1)}(2dl)^{2m(m_0+1)}$.

Proof. If $x \in \mathrm{int}\, E$, by Cramer's Rule and Hadamard's inequality we can write

$$\|(d_x f)^{-1}\| \leq \text{const.}\,|\det\ d_x f|^{-1}.$$

Let $X = \{x \in \mathbb{R}^n :\ \det\ d_x f = 0\}$. By Łojasiewicz's inequality, there exist constants $A > 0$ and $\alpha \geq 1$ such that for each $x \in \mathrm{int}\, E$ we have

$$|\det d_x f| \geq A[\mathrm{dist}(x, X)]^\alpha \geq A[\mathrm{dist}(x, \partial E)]^\alpha.$$

This completes the proof of the first part of the theorem. To prove (2.8.2) fix a polynomial $p \in \mathscr{P}_k$ and choose again a point $x \in \operatorname{int} E$. By [B2, Corollary 3.5] E satisfies $M(2m)$ with the constant $M_1 = \frac{\sqrt{2}}{M}(2d)^{2m}$, whence by (2.8.1) and Corollary 2.3 we can write

$$\begin{aligned}\|\operatorname{grad} p(f(x))\| &\le \|(d_x f)^{-1}\| \|\operatorname{grad} (p \circ f)(x)\| \\ &\le M_0[\operatorname{dist}(x, \partial E)]^{-m_0}\sqrt{2}M^{-1}(2dlk)^{2m}\|p\|_{f(E)}.\end{aligned}$$

By the assumptions, E admits a parametrization $h : E \times [0, 1) \ni (x, t) \to h(x, t) \in \operatorname{int} E$ such that for each x, $h(x, 1) = x$, $h(x, \cdot)$ is a polynomial of degree at most d and $\operatorname{dist}(h(x, t), \partial E) \ge M(1-t)^m$. Hence we obtain, for any $v \in \mathbb{R}^n$ with $\|v\| = 1$,

$$|D_v p(f(h(x, t^2)))| \le LM^{-mm_0}(1-t^2)^{-mm_0}$$

with $L = M_0\sqrt{2}M^{-1}(2dlk)^{2m}\|p\|_{f(E)}$. By a generalization of Schur's theorem (see [B2, Lemma 2.4]), this implies that

$$|D_v p(f(h(x, t^2)))| \le LM^{-mm_0}(2dlk)^{2mm_0}$$

for $t \in [-1, 1]$, whence

$$\|\operatorname{grad} p(f(h(x, t^2)))\| \le \sqrt{2}LM^{-mm_0}(2dlk)^{2mm_0}$$

for $t \in [-1, 1]$. Hence by setting $t^2 = 1$ we get the assertion (2.8.2) of the theorem.

The proof of Theorem 2.8 yields the estimate

$$\mu(f(E)) \le \mu(E) + 2mm_0,$$

which is sharp in the following sense.

Example 2.9. Take E to be the set $\{x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \ge 0, x_1^2 + x_2 \le 1\}$. Since E is convex, it satisfies $M(r)$ with $r = 2$. Let $f(x_1, x_2) = (x_1^2, x_2^p)$, $p \in \mathbb{N}$, $p \ge 2$. Then we have, for $x \in \operatorname{int} E$,

$$\|(d_x f)^{-1}\| \le \frac{1}{2}(\min\{x_1, x_2\})^{-(p-1)} \le \frac{1}{2}(\operatorname{dist}(x, \partial E))^{-(p-1)}.$$

Thus, we have $m = 1$ and $m_0 = p - 1$, and by Theorem 2.8, $\mu(f(E)) \le 2p$. On the other hand, since $f(E) = \{x \in \mathbb{R}^2 : x_1, x_2 \ge 0, x_1 + \sqrt[p]{x_2} \le 1\}$, by [G], $\mu(f(E)) = 2p$.

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JAGIELLONIAN UNIVERSITY OF CRACOW, INSTITUTE OF MATHEMATICS, REYMONTA 4, 30-059 KRAKÓW, POLAND

E-mail address, M. Baran: baran@im.uj.edu.pl

E-mail address, W. Pleśniak: plesniak@im.uj.edu.pl